Analysis III: Auditorium Exercise-03

For Engineering Students

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Consider two functions

$$f, g : D \subset \mathbb{R}^n \to \mathbb{R}^m,$$

where D is open, and $x_0 \in D$. If f and g are differentiable at x_0, then the linear combination $\alpha f + \beta g$ with $\alpha, \beta \in \mathbb{R}$ is also differentiable at x_0.

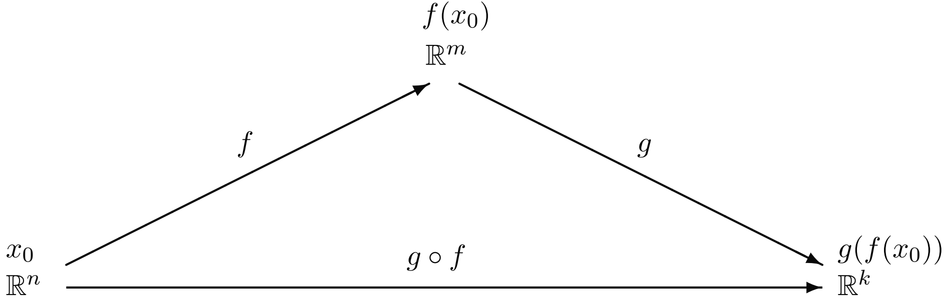
For the Jacobian matrix of the linear combination, we have

$$J(\alpha f + \beta g)(x_0) = \alpha J f(x_0) + \beta J g(x_0)$$

Let's consider a function $f : D \subset \mathbb{R}^n \to \mathbb{R}^m$ and a function $g : E \subset \mathbb{R}^m \to \mathbb{R}^k$.

For $f(D) \subset E$, the composition of f and g is defined as

$$g \circ f : D \to \mathbb{R}^k\,, \quad (g \circ f)(x) := g(f(x))$$


$f(x_0)$
$\mathbb{R}^m$
f
g
x_0
$\mathbb{R}^n$
$g \circ f$
$g(f(x_0))$
$\mathbb{R}^k$

Theorem: Chain Rule

If f is totally differentiable at x^0 and g is totally differentiable at $y^0 := f(x^0)$, then $g \circ f$ is totally differentiable at x^0 and the following holds

$$J(g \circ f)(x^0) = Jg(f(x^0)) \cdot Jf(x^0)$$

Example 01

Let,

$$w : \mathbb{R}^2 \stackrel{\Phi}{\to} \mathbb{R}^2 \stackrel{\tilde{w}}{\to} \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} \mapsto \tilde{w}(u,v) = w(x,y)$$

$$\begin{aligned} Jw &= (w_x, w_y) = J(\tilde{w} \circ \Phi) = J\tilde{w} \cdot J\Phi \\ &= (\tilde{w}_u, \tilde{w}_v) \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = (\tilde{w}_u u_x + \tilde{w}_v v_x \ , \ \tilde{w}_u u_y + \tilde{w}_v v_y) \end{aligned}$$

Example 02

Calculate the Jacobi matrix using the chain rule and directly:

$$f : \mathbb{R}^2 \quad \overset{f_1}{\rightarrow} \quad \mathbb{R}^2 \quad \overset{f_2}{\rightarrow} \quad \mathbb{R}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} r = ye^x \\ s = x^3 \end{pmatrix} \mapsto r\cos(s^2)$$

Example: Chain rule

$$Jf_1(x,y) = \begin{pmatrix} ye^x & e^x \\ 3x^2 & 0 \end{pmatrix},$$

$$Jf_2(r,s) = \left(\ \cos(s^2)\,, \quad -2rs\sin(s^2)\ \right)$$

$$\begin{aligned} Jf(x,y) &= J(f_2 \circ f_1)(x,y) = Jf_2(f_1(x,y)) \cdot Jf_1(x,y) \\ &= \left(\ \cos((x^3)^2)\,, \quad -2ye^x x^3 \sin((x^3)^2)\ \right) \cdot \begin{pmatrix} ye^x & e^x \\ 3x^2 & 0 \end{pmatrix} \\ &= \left(\ ye^x\cos(x^6) - 6x^5ye^x\sin(x^6)\,, \quad e^x\cos(x^6)\ \right) \end{aligned}$$

Example: Direct

$$f_2(f_1(x,y)) = f_2(r(x,y)), s(x,y)) = f(x,y) = ye^x \cos(x^6)$$

$$\Rightarrow \quad Jf(x,y) = \big(\; ye^x \cos(x^6) - 6x^5 ye^x \sin(x^6), \quad e^x \cos(x^6) \;\big)$$

Calculate the Jacobi matrix using the chain rule and directly:

$$g : \mathbb{R}^3 \quad \overset{g_1}{\to} \quad \mathbb{R}^2 \quad \overset{g_2}{\to} \quad \mathbb{R}^3$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} u = \sin(2yz) \\ v = x^2 \end{pmatrix} \mapsto \begin{pmatrix} 2v - 3u \\ e^{2u+v} \\ u^3 v \end{pmatrix}.$$

Definition:
Let Φ be a C^1 function, and $U, V \subset \mathbb{R}^n$ be open sets, with

$$\Phi : U \to V \quad \text{and} \quad u \mapsto \Phi(u)$$

Here, $u = (u_1, u_2, \cdots, u_n)^T$ and $\Phi(u) = (\Phi_1(u), \Phi_2(u), \cdots, \Phi_n(u))^T$.

The Jacobian matrix $J\Phi(u^0)$ is assumed to be regular for every $u^0 \in U$, and there exists a C^1 inverse function $\Phi^{-1} : V \to U$. Then, $x = \Phi(u)$ is referred to as a **coordinate transformation** from the coordinates u to the coordinates x.

Polar Coordinates

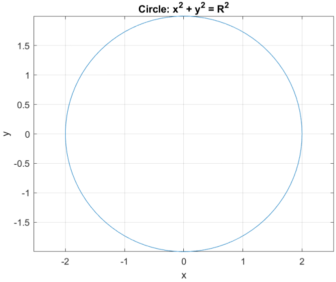
Let $u = (r, \varphi)^T$
with $0 < r$ and $-\pi < \varphi < \pi$

$$x = \begin{pmatrix} x \\ y \end{pmatrix} = \Phi(r, \varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \end{pmatrix}$$

The equation for circle is:

$$x^2 + y^2 = R^2$$

describes the boundary K of a circular disk with a radius of R and a center at $(0, 0)$.
K can be represented using Polar coordinates with $R = r$.

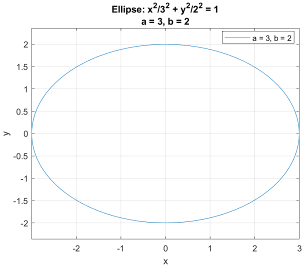


Ellipse Equation

The equation for ellipse is:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

describes the boundary E of an ellipse with the semi-axes a and b and a center at $(0, 0)$.
E can be represented as $(x, y) = (a\cos(\varphi), b\sin(\varphi))$.



In cylindrical coordinates, a point is represented as $u = (r, \varphi, z)^T$ with $0 < r, -\pi < \varphi < \pi, z \in \mathbb{R}$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, z) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

Spherical Coordinates

In spherical coordinates, a point is represented as $u = (r, \varphi, \theta)^T$ with $0 < r, -\pi < \varphi < \pi, -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, \theta) = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

The inequality

$$x^2 + y^2 + z^2 \leq R^2$$

describes a **Solid Sphere** K with a radius of R and a center at $(0, 0, 0)$.

With $0 \leq r \leq R$, K can be represented using spherical coordinates.

$$x^2 + y^2 = 3$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:

Circle: Radius $r = \sqrt{3}$, Center $(0, 0)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (\sqrt{3}\cos(\varphi), \sqrt{3}\sin(\varphi))$$

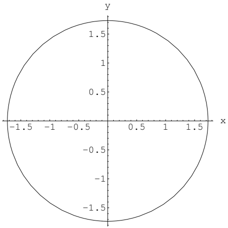


Figure: Circle $x^2 + y^2 = 3$

$$4x^2 + 9y^2 = 36$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:

Ellipse: Semi-axes $a = 3$ and $b = 2$, Center $(0, 0)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (3\cos(\varphi), 2\sin(\varphi))$$

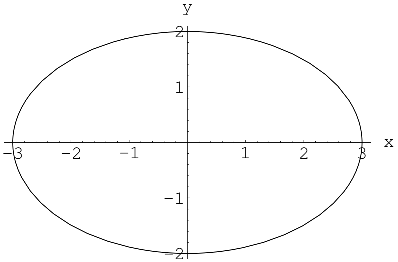


Figure: Ellipse $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1$

$$16x^2 + 3y^2 + 6y + 3 = 48$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:
Through completing the square, we obtain

$$16x^2 + 3y^2 + 6y + 3 = 16x^2 + 3(y+1)^2 = 48$$
$$\Leftrightarrow \quad \frac{x^2}{(\sqrt{3})^2} + \frac{(y+1)^2}{4^2} = 1$$

Ellipse: Semi-axes $a = \sqrt{3}$ and $b = 4$, Center $(0, -1)$
Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (\sqrt{3}\cos(\varphi), 4\sin(\varphi) - 1)$$

Example 05

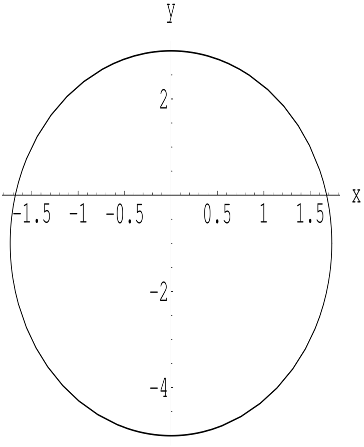


Figure: Ellipse $\dfrac{x^2}{(\sqrt{3})^2} + \dfrac{(y+1)^2}{4^2} = 1$

Example 06

$$x^2 - 6x + 9 + y^2 = 25$$

Draw the circle or ellipse and represent the solution sets of the equation using polar coordinates, and display the (x, y) coordinates.

Solution:
Through completing the square, we obtain

$x^2 - 6x + 9 + y^2 = (x-3)^2 + y^2 = 5^2.$
Circle: Radius $r = 5$, Center $(3, 0)$

Representation using Polar Coordinates with $-\pi \leq \varphi < \pi$

$$(x, y) = (5\cos(\varphi) + 3, 5\sin(\varphi))$$

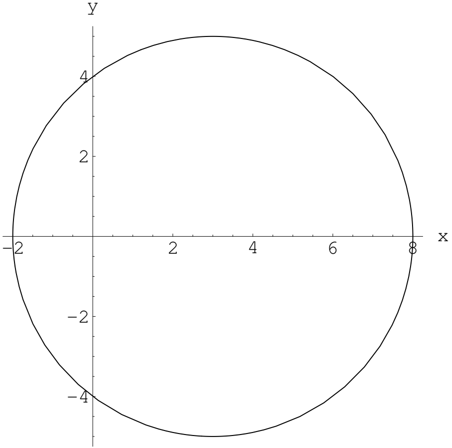


Figure: Circle $(x-3)^2 + y^2 = 5^2$

Draw the solution sets of the following region in $\mathbb{R}^3$ and represent them using cylindrical coordinates. $x^2 + y^2 \leq 4$ with $x \leq 0$ and $1 \leq z \leq 3$

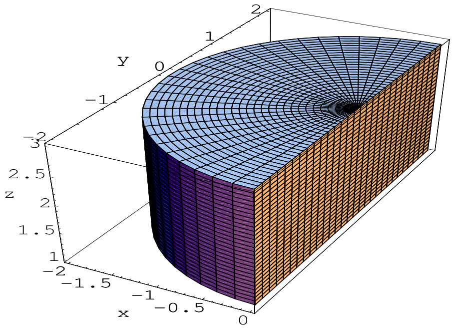


Figure: Half cylinder Z

Example 07

Cylindrical Coordinates for Z: $\quad u = (r, \varphi, z)^T$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix} = (r, \varphi, z)$$

with $\quad 0 \le r \le 2\,, \quad \dfrac{\pi}{2} \le \varphi \le \dfrac{3\pi}{2}\,, \quad 1 \le z \le 3$

Draw the solution sets of the following region in $\mathbb{R}^3$ and represent them using spherical coordinates. $x^2 + y^2 + z^2 \leq 16$, $0 \leq y$

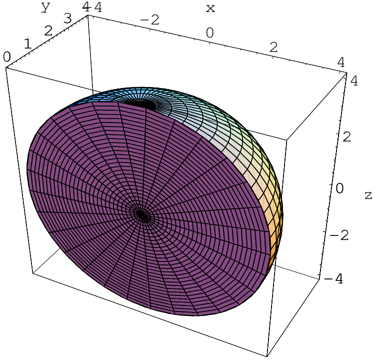


Figure: Half sphere H

Spherical Coordinates for H: $\quad u = (r, \varphi, \theta)^T$

$$x = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, \theta) = \begin{pmatrix} r\cos(\varphi)\cos(\theta) \\ r\sin(\varphi)\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$$

with $\quad 0 \leq r \leq 4\,, \quad 0 \leq \varphi \leq \pi\,, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$

Consider the coordinate transformation

$$\Phi(x, y) = \begin{pmatrix} u(x, y) \\ v(x, y) \end{pmatrix} = \begin{pmatrix} x - y \\ x + y \end{pmatrix}$$

with $(x, y) \in Q := [-1, 1] \times [-1, 1]$.

- Calculate $J\Phi(x, y)$ and $\det(J\Phi(x, y))$.
- Calculate $\Phi^{-1}(u, v), J\Phi^{-1}(u, v),\ \det(J\Phi^{-1}(u, v))$.
- Draw Q and $\Phi(Q)$

Example 09(i)

$$\Phi(x,y) = \begin{pmatrix} u(x,y) \\ v(x,y) \end{pmatrix} = \begin{pmatrix} x-y \\ x+y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

This is a linear transformation, more precisely, it's a rotation and scaling by 45° with a factor of $\sqrt{2}$, as:

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \sqrt{2} \begin{pmatrix} \cos(45^\circ) & -\sin(45^\circ) \\ \sin(45^\circ) & \cos(45^\circ) \end{pmatrix}.$$

$$J\Phi(x,y) = \begin{pmatrix} u_x & u_y \\ v_x & v_y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix},$$

$\det(J\Phi(x,y)) = 2$

$$
\begin{aligned}
\Phi^{-1}(u,v) &= \begin{pmatrix} x(u,v) \\ y(u,v) \end{pmatrix} \\
&= \begin{pmatrix} (u+v)/2 \\ (v-u)/2 \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}, \\
J\Phi^{-1}(u,v) &= \begin{pmatrix} x_u & x_v \\ y_u & y_v \end{pmatrix} \\
&= \begin{pmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{pmatrix} = (J\Phi)^{-1},
\end{aligned}
$$

$$\det(J\Phi^{-1}(u,v)) = 1/2$$

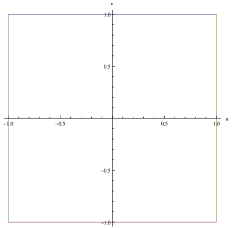


Figure: Q

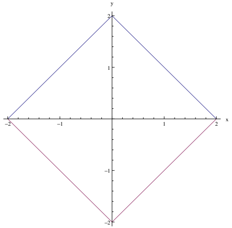


Figure: $\Phi(Q)$

THANK YOU