Analysis III: Auditorium Exercise-02

For Engineering Students

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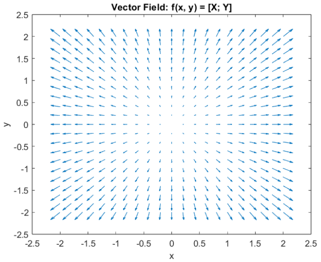
Topics for today

- Vector Field
- Jacobian Matrix
- Streamlines
- Directional derivative
- Vector Operators

Let D be open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is called a **vector field** on D, if every function $f_i(x)$ of $f = (f_1, ..., f_n)^T$ is a $C^{\mathcal{K}}$-function, then f is called $c^{\mathcal{K}}$-vector field.



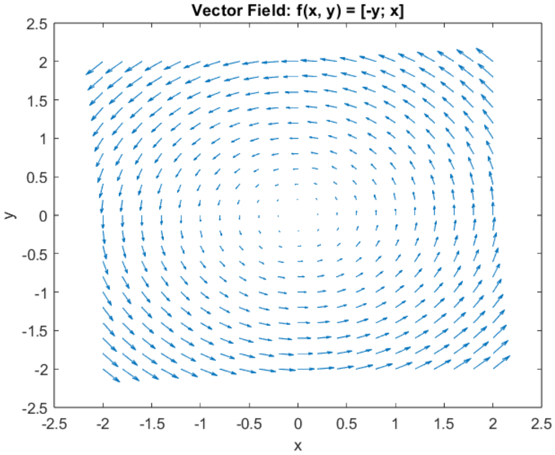
$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Example

Find the vector field in the xy -plane

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Example


Vector Field: f(x, y) = [-y; x]
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
y
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
x

Let $f : D \to \mathbb{R}^m, D \subset \mathbb{R}^n$, $x = (x_1, ..., x_n)^T \in D$,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, ..., x_n) \\ f_2(x_1, ..., x_n) \\ \vdots \\ f_m(x_1, ..., x_n) \end{pmatrix}$$

Then the Jacobian Matrix is $m \times n$ matrix $J_{ij} = \frac{\partial f_i}{\partial x_j}(x)$:

$$Jf(x) = \begin{pmatrix} \text{grad } f_1(x) \\ \text{grad } f_2(x) \\ \vdots \\ \text{grad } f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \dots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

Determinant of a Matrix

A matrix is given

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

The determinant is

$$det\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right) = 2$$

Jacobian Determinant

- If $m = n$ the determinant of the Jacobian is knownn as the Jacobian determinant of f
- The Jacobian is used when making a change of variables and a coordinate transformation.

Compute the Jacobian matrix and the Jacobian determinant of the following vector function:

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x + x^2 y \\ 5x + sin(y) \end{pmatrix}$$

$$J(x,y) = \begin{pmatrix} \partial f_1/\partial x & \partial f_1/\partial y \\ \partial f_2/\partial x & \partial f_2/\partial y \end{pmatrix} = \begin{pmatrix} 1+2xy & x^2 \\ 5 & \cos y \end{pmatrix}$$

$$\det \begin{vmatrix} 1+2xy & x^2 \\ 5 & \cos(y) \end{vmatrix} = (1+2xy)\cos(y) - 5x^2 = \cos(y) + 2xy\cos(y) - 5x^2$$

Exercise 2

Compute the Jacobian matrix of the following vector function:

$$f(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{pmatrix} = \begin{pmatrix} xe^y + x^2 z \\ e^{x^2+2y^2} \end{pmatrix}$$

$m \neq n$

$$J(x,y,z) = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix} = \begin{pmatrix} e^y + 2xz & xe^y & x^2 \\ 2xe^{x^2+2y^2} & 4ye^{x^2+2y^2} & 0 \end{pmatrix}$$

Chain rule For vector functions

Let $f : D \to \mathbb{R}^m$ be differentiable in $x^0 \in D$, D is open. Let $g : E \to \mathbb{R}^k$ be differentiable in $y^0 = f(x^0 \in E \subset \mathbb{R}^m)$, E is open. Then $g \circ f$ is differentiable in x^0.
For the differentials it holds

$$\mathrm{d}(g \circ f)(x^0) = \mathrm{d}g(y^0) \circ \mathrm{d}f(x^0)$$

and analogously for the Jacobian matrix

$$\mathrm{J}(g \circ f)(x^0) = \mathrm{J}g(y^0) \circ \mathrm{J}f(x^0)$$

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ be a vector valued function of one variable defined as follows

$$f(x, y, z) = e^z cos(2x) sin(3y)$$

$$g(t) = (x(t), y(t), z(t)) = (2t, t^2, t^3)$$

Compute the derivative of the composition $f \circ g$.

$$\frac{\partial f}{\partial x} = -2e^z \sin(2x)\sin(3y)$$

$$\frac{\partial f}{\partial y} = 3e^z \cos(2x)\cos(3y)$$

$$\frac{\partial f}{\partial z} = e^z \cos(2x)\sin(3y)$$

$$\frac{\partial}{\partial t}(f \circ g) = \text{grad } f\big(x(t), y(t), z(t)\big) \cdot g'(t)$$

$$= 2e^z \sin(2x)\sin(3y)\cdot 2 + 3e^z \cos(2x)\cos(3y)\cdot 2t + e^z\cos(2x)\sin(3y)\cdot 3t^2$$

as: $x(t) = 2t$, $y(t) = t^2$, $z(t) = t^3$

$$\frac{dx(t)}{dt} = 2 \qquad \frac{dy(t)}{dt} = 2t \qquad \frac{dz(t)}{dt} = 3t^2$$

$$= 2e^{t^3}\sin(2\cdot 2t)\sin(3\cdot t^2) + 3e^{t^3}\cos(2\cdot 2t)\cos(3\cdot t^2)\cdot 2t + e^{t^3}\cos(2\cdot 2t)\sin(3\cdot t^2)\cdot 3t^2$$

Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = x^2y + xy^2$$

$$\frac{\partial f}{\partial x} = 2xy + y^2$$

$$\frac{\partial f}{\partial y} = x^2 + 2xy$$

and $g : \mathbb{R}^2 \to \mathbb{R}^2$

$$g(s,t) = \begin{pmatrix} x(s,t) \\ y(s,t) \end{pmatrix} = \begin{pmatrix} 2s + t \\ s - 2t \end{pmatrix}$$

$$\frac{\partial g}{\partial s} = \begin{pmatrix} \partial x/\partial s \\ \partial y/\partial s \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$\frac{\partial g}{\partial t} = \begin{pmatrix} \partial x/\partial t \\ \partial y/\partial t \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Compute the derivative of the composition $f \circ g$.

$$\frac{\partial}{\partial t}(f \circ g)(s,t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial t} = \text{- - - -}$$

$$\frac{\partial}{\partial s}(f \circ g)(s,t) = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} = \text{- - - -}$$

Streamlines

Let $u = (u_1(x, y), u_2(x, y))^T$ be a velocity field of the two dimensional flow. The streamlines associated the flow u are the solution of the system of the differential equations

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \end{cases}$$

or the differential equation

$$y'(x) = \frac{u_2(x, y)}{u_1(x, y)}$$

Calculate the streamline passing through a point $(x_0, y_0)^T$ for the stagnation point flow

$$u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

we have to solve the diff. eq^n $y'(x) = \frac{u_2(x,y)}{u_1(x,y)} \Rightarrow \frac{dy}{dx} = \frac{u_2(x,y)}{u_1(x,y)} \Rightarrow \frac{dy}{dx} = \frac{-y}{x}$

now have to integrate to find the streamline, $\int \frac{dx}{x} = -\int \frac{dy}{y} \Rightarrow \ln|x| = -\ln|y| + c$

$\Rightarrow e^{\ln|x||y|} = e^c = A \rightarrow$ arbitrary constant

$\Rightarrow |x||y| = A$

using $(x_0, y_0)^T$ to find the constant $A \rightarrow |x_0||y_0| = A$

Now, $|x||y| = |x_0||y_0| \Rightarrow |y| = \frac{|x_0||y_0|}{|x|} \rightarrow$ eq^n of streamline.

Exercise 6

Calculate the streamline passing through a point $(0,0)^T$ for the stagnation point flow

$$u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} x \\ x(x-1)(y+1) \end{pmatrix}$$

$$\frac{dy}{dx} = \frac{x(x-1)(y+1)}{x} = (x-1)(y+1)$$

$$\Rightarrow \int \frac{dy}{y+1} = \int (x-1)\,dx \Rightarrow \ln|y+1| + c_1 = \frac{x^2}{2} - x + c_2$$

$$\Rightarrow \ln|y+1| = \frac{x^2}{2} - x + \underbrace{c_2 - c_1}_{\to\, c} \text{ (constant)}$$

$$\Rightarrow e^{\ln|y+1|} = e^{x^2/2 - x} \cdot e^{c}$$

$$\Rightarrow |y+1| = e^{x^2/2 - x} \cdot A \to \text{constant}$$

for A we use $(0,0)^T$

$$A = \frac{|y+1|}{e^{x^2/2 - x}} = 1$$

$$\text{So, } |y+1| = e^{x^2/2 - x} \Rightarrow y = e^{x^2/2 - x} - 1 \to \text{eq}^n \text{ of streamline}$$

Let $f : D \to \mathbb{R}$,$D \subset \mathbb{R}^n$, D is open , $x^0 \in D$, $v \in \mathbb{R}^n \backslash 0$ a vector. Then

$$D_v f(x^0) := \lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x^0)}{t}$$

is called the directional derivative of $f(x)$ in the direction v.

Exercise 7

Calculate by definition the directional derivative of the function $f(x_1, x_2) = 2x_1 + x_1x_2$ at a point (x_1^0, x_2^0) in the direction $v = (v_1, v_2)^T$.

$$D_v f(x^0) = \lim_{t \to 0^+} \frac{f(x^0 + tv) - f(x^0)}{t}$$

substituting the function $f(x_1, x_2)$ into the defination.

$$D_v f(x^0) = \lim_{t \to 0^+} \frac{\left(2(x_1^0 + tv_1) + (x_1^0 + tv_1)(x_2^0 + tv_2)\right) - (2x_1^0 + x_1^0 x_2^0)}{t}$$

$$= 2v_1 + x_1^0 v_2 + v_1 x_2^0 + \lim_{t \to 0} \frac{t^2 v_1 v_2}{t}$$

$$= 2v_1 + x_1^0 v_2 + v_1 x_2^0 + \lim_{t \to 0} t (v_1 v_2)$$

$$= 2v_1 + x_1^0 v_2 + x_2^0 v_1$$

Exercise 8

Let $f(x, y) = x^2 y$. Now compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(1, 2)$ at the point $(3, 2)$

$$\text{grad } f(3,2) = (2xy, x^2)\Big|_{(3,2)} = (12, 9)$$

$$D_v f(x,y) = \text{grad } f(x,y) \cdot v$$

$$D_{(1,2)} f(3,2) = \text{grad } f(3,2) \cdot v$$

unit vector, $v = \frac{(1,2)}{\|(1,2)\|} = \frac{(1,2)}{\sqrt{1^2+2^2}} = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$

$$= (12, 9)\begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix} = \frac{12}{\sqrt{5}} + \frac{18}{\sqrt{5}} = \frac{30}{\sqrt{5}}$$

Exercise 9

Let $f(x, y) = x^2 y$. Now compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(2, 1)$ at the point $(3, 2)$

Compute $D_v f(x,y)$ for $f(x,y) = cos(\frac{x}{y})$ in the direction $v = (3,-4)$

$$D_v f(x,y) = \text{grad}\, f(x,y) \cdot v$$

$$= \left(-\frac{1}{y}\sin\left(\frac{x}{y}\right), \frac{x}{y^2}\sin\left(\frac{x}{y}\right)\right) \cdot \begin{pmatrix} 3/5 \\ -4/5 \end{pmatrix}$$

$$= -\frac{3}{5y}\sin\left(\frac{x}{y}\right) - \frac{4x}{5y^2}\sin\left(\frac{x}{y}\right)$$

$$\text{grad}\, f(x,y) = \left(-\frac{1}{y}\sin\left(\frac{x}{y}\right), \frac{x}{y^2}\sin\left(\frac{x}{y}\right)\right)$$

unit vector: $v = \frac{(3,-4)}{\sqrt{3^2+(-4)^2}} = \left(\frac{3}{5}, \frac{-4}{5}\right)$

Exercise 11

Compute $D_v f(3,-1,0)$ for $f(x,y,z) = 4x - y^2 e^{3xz}$ in the direction $v = (-1,4,2)$. Is it a direction of descent or ascent?

$D_v f(3,-1,0) = \text{grad } f(3,-1,0) \cdot v$

$\text{grad } f(3,-1,0) = \left(4 - 3z y^2 e^{3xz}, -2y e^{3xz}, -3x y^2 e^{3xz}\right)\Big|_{(3,-1,0)}$

$=$

if $D_v f > 0 \rightarrow$ ascent

$D_v f < 0 \rightarrow$ descent

For the vectorfield $= (f_1, \ldots, f_n)^T$ in $\mathbb{R}^n$ and $x = (x_1, ...x_n)$ then the divergence of the vector field is

$$\text{div } f := \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

- The divergence of the vector field f is a scalar field.

Example

Compute the divergence of the vector field:

- $\vec{f} = x\hat{i} + y\hat{j}$
- $\vec{f} = -x\hat{i} - y\hat{j}$
- $\vec{f} = -y\hat{i} + x\hat{j}$

Let f be the in $\mathbb{R}^3$ and the rotation of f or the curl of f:

$$f(x_1, x_2, x_3) = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix}, \quad \text{rot } f := \begin{pmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{pmatrix}$$

- The curl of a vector field $\vec{f}$ is indeed a vector field. The curl operation takes a vector field as input and produces another vector field as its output.

Example

Compute the divergence of the vector field:

- $\vec{f} = x\hat{i} + y\hat{j}$
- $\vec{f} = -y\hat{i} + x\hat{j}$

Exercise 12

Compute the div (f) and rot (f) for $f(x,y,z) = \begin{pmatrix} x^2y \\ z^3 - 3x \\ 4y^2 \end{pmatrix} \begin{matrix} \to f_1 \\ \to f_2 \\ \to f_3 \end{matrix}$

$$\operatorname{div} f = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} = 2xy + 0 + 0 = 2xy \quad (\operatorname{div} f = \nabla \cdot f)$$

$$\operatorname{rot} f = \nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = i \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix} - j \begin{vmatrix} \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\ f_3 & f_1 \end{vmatrix} + k \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix}$$

$$= i\left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right) - j\left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right) + k\left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)$$

$$= i(8y - 3z^2) - j(0 - 0) + k(3 - x^2) = \begin{pmatrix} 8y - 3z^2 \\ 0 \\ 3 - x^2 \end{pmatrix}$$

Exercise 13

Compute the div (f) and rot (f) for $f(x,y,z) = 2x^2z\vec{i} + yz\vec{j} + \vec{k}$.

$f_1 = 2x^2z,\quad f_2 = yz,\quad f_3 = 1 \rightarrow \begin{pmatrix} 2x^2z \\ yz \\ 1 \end{pmatrix}$

$$\text{rot } f = \nabla \times f = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 - y \\ 2x^2 - 0 \\ 0 - 0 \end{pmatrix} = \begin{pmatrix} -y \\ 2x^2 \\ 0 \end{pmatrix}$$

Compute the Jacobian matrix of the following vector function:

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \\ f_3(x,y) \end{pmatrix} = \begin{pmatrix} sin(y) \\ x^3 + cos(x) \\ x^2y^2 \end{pmatrix}$$

→ follow exercise 2.

THANK YOU