Analysis III: Auditorium Exercise-02

For Engineering Students

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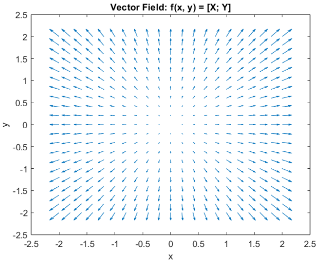
Topics for today

- Vector Field
- Jacobian Matrix
- Streamlines
- Directional derivative
- Vector Operators

Vector Field

Let D be open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is called a **vector field** on D, if every function $f_i(x)$ of $f = (f_1, ..., f_n)^T$ is a $C^{\mathcal{K}}$-function, then f is called $c^{\mathcal{K}}$-vector field.



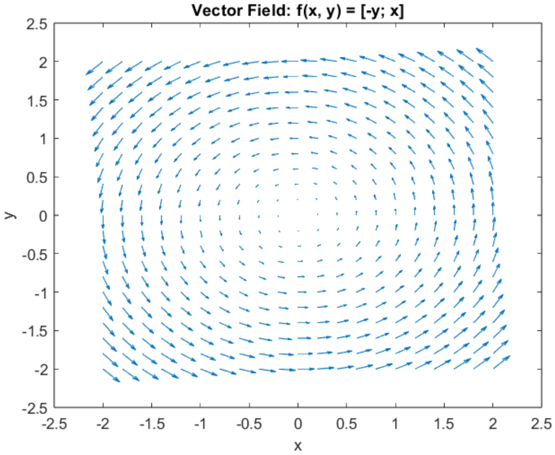
$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

Example

Find the vector field in the xy -plane

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$$

Example


Vector Field: f(x, y) = [-y; x]
2.5
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
-2.5
y
-2.5
-2
-1.5
-1
-0.5
0
0.5
1
1.5
2
2.5
x

Let $f : D \to \mathbb{R}^m, D \subset \mathbb{R}^n$, $x = (x_1, ..., x_n)^T \in D$,

$$f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \\ \vdots \\ f_m(x) \end{pmatrix} = \begin{pmatrix} f_1(x_1, ..., x_n) \\ f_2(x_1, ..., x_n) \\ \vdots \\ f_m(x_1, ..., x_n) \end{pmatrix}$$

Then the Jacobian Matrix is $m \times n$ matrix $J_{ij} = \frac{\partial f_i}{\partial x_j}(x)$:

$$Jf(x) = \begin{pmatrix} \text{grad } f_1(x) \\ \text{grad } f_2(x) \\ \vdots \\ \text{grad } f_m(x) \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1}(x) & \frac{\partial f_1}{\partial x_2}(x) \dots & \frac{\partial f_1}{\partial x_n}(x) \\ \frac{\partial f_2}{\partial x_1}(x) & \frac{\partial f_2}{\partial x_2}(x) \dots & \frac{\partial f_2}{\partial x_n}(x) \\ \vdots & & \\ \frac{\partial f_m}{\partial x_1}(x) & \frac{\partial f_m}{\partial x_2}(x) \dots & \frac{\partial f_m}{\partial x_n}(x) \end{pmatrix}$$

Determinant of a Matrix

A matrix is given

$$M = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

The determinant is

$$det\left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}\right) = 2$$

Jacobian Determinant

- If $m = n$ the determinant of the Jacobian is knownn as the Jacobian determinant of f
- The Jacobian is used when making a change of variables and a coordinate transformation.

Compute the Jacobian matrix and the Jacobian determinant of the following vector function:

$$f(x,y) = \begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix} = \begin{pmatrix} x + x^2 y \\ 5x + sin(y) \end{pmatrix}$$

Compute the Jacobian matrix of the following vector function:

$$f(x,y,z) = \begin{pmatrix} f_1(x,y,z) \\ f_2(x,y,z) \end{pmatrix} = \begin{pmatrix} xe^y + x^2 z \\ e^{x^2+2y^2} \end{pmatrix}$$

Chain rule For vector functions

Let $f : D \to \mathbb{R}^m$ be differentiable in $x^0 \in D$, D is open. Let $g : E \to \mathbb{R}^k$ be differentiable in $y^0 = f(x^0 \in E \subset \mathbb{R}^m)$, E is open.
Then $g \circ f$ is differentiable in x^0.
For the differentials it holds

$$\mathrm{d}(g \circ f)(x^0) = \mathrm{d}g(y^0) \circ \mathrm{d}f(x^0)$$

and analogously for the Jacobian matrix

$$\mathrm{J}(g \circ f)(x^0) = \mathrm{J}g(y^0) \circ \mathrm{J}f(x^0)$$

Exercise 3

Let $f : \mathbb{R}^3 \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^3$ be a vector valued function of one variable defined as follows

$$f(x, y, z) = e^z cos(2x) sin(3y)$$

$$g(t) = (x(t), y(t), z(t)) = (2t, t^2, t^3)$$

Compute the derivative of the composition $f \circ g$.

Exercise 4

Let $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x, y) = x^2 y + x y^2$$

and $g : \mathbb{R}^2 \to \mathbb{R}^2$

$$g(s, t) = \begin{pmatrix} x(s, t) \\ y(s, t) \end{pmatrix} = \begin{pmatrix} 2s + t \\ s - 2t \end{pmatrix}$$

Compute the derivative of the composition $f \circ g$.

Streamlines

Let $u = (u_1(x,y), u_2(x,y))^T$ be a velocity field of the two dimensional flow. The streamlines associated the flow u are the solution of the system of the differential equations

$$\begin{cases} \dot{x} = u_1 \\ \dot{y} = u_2 \end{cases}$$

or the differential equation

$$y'(x) = \frac{u_2(x,y)}{u_1(x,y)}$$

Calculate the streamline passing through a point $(x_0, y_0)^T$ for the stagnation point flow

$$u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

Calculate the streamline passing through a point $(0,0)^T$ for the stagnation point flow

$$u = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix} = \begin{pmatrix} x \\ x(x-1)(y+1) \end{pmatrix}$$

Let $f : D \to \mathbb{R}$,$D \subset \mathbb{R}^n$, D is open , $x^0 \in D$, $v \in \mathbb{R}^n \backslash 0$ a vector. Then

$$D_v f(x^0) := \lim_{t \to 0^+} \frac{f(x_0 + tv) - f(x^0)}{t}$$

is called the directional derivative of $f(x)$ in the direction v.

Exercise 7

Calculate by definition the directional derivative of the function $f(x_1, x_2) = 2x_1 + x_1x_2$ at a point (x_1^0, x_2^0) in the direction $v = (v_1, v_2)^T$.

Exercise 8

Let $f(x, y) = x^2 y$. Now compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(1, 2)$ at the point $(3, 2)$

Exercise 9

Let $f(x, y) = x^2 y$. Now compute

- grad $f(3, 2)$
- the derivative of f in the direction of $(2, 1)$ at the point $(3, 2)$

Exercise 10

Compute $D_v f(x,y)$ for $f(x,y) = cos(\frac{x}{y})$ in the direction $v = (3,-4)$

Exercise 11

Compute $D_v f(3,-1,0)$ for $f(x,y,z) = 4x - y^2 e^{3xz}$ in the direction $v = (-1,4,2)$. Is it a direction of descent or ascent?

Divergence

For the vectorfield $= (f_1, \ldots, f_n)^T$ in $\mathbb{R}^n$ and $x = (x_1, ...x_n)$ then the divergence of the vector field is

$$\text{div } f := \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_i}$$

- The divergence of the vector field f is a scalar field.

Example

Compute the divergence of the vector field:

- $\vec{f} = x\hat{i} + y\hat{j}$
- $\vec{f} = -x\hat{i} - y\hat{j}$
- $\vec{f} = -y\hat{i} + x\hat{j}$

Rotation

Let f be the in $\mathbb{R}^3$ and the rotation of f or the curl of f:

$$f(x_1, x_2, x_3) = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \end{pmatrix}, \quad \text{rot } f := \begin{pmatrix} \frac{\partial f_3}{\partial x_2} - \frac{\partial f_2}{\partial x_3} \\ \\ \frac{\partial f_1}{\partial x_3} - \frac{\partial f_3}{\partial x_1} \\ \\ \frac{\partial f_2}{\partial x_1} - \frac{\partial f_1}{\partial x_2} \end{pmatrix}$$

- The curl of a vector field $\vec{f}$ is indeed a vector field. The curl operation takes a vector field as input and produces another vector field as its output.

Compute the divergence of the vector field:

- $\vec{f} = x\hat{i} + y\hat{j}$
- $\vec{f} = -y\hat{i} + x\hat{j}$

Compute the div (f) and rot (f) for $f(x, y, z) = \begin{pmatrix} x^2 y \\ z^3 - 3x \\ 4y^2 \end{pmatrix}$

Exercise 13

Compute the div (f) and rot (f) for $f(x, y, z) = 2x^2 z\vec{i} + yz\vec{j} + \vec{k}$.

Additional Exercise

Compute the Jacobian matrix of the following vector function:

$$f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \\ f_3(x, y) \end{pmatrix} = \begin{pmatrix} sin(y) \\ x^3 + cos(x) \\ x^2y^2 \end{pmatrix}$$

THANK YOU