Analysis III: Auditorium Exercise-01

For Engineering Students

Md Tanvir Hassan
University of Hamburg

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Office Hours

1. Mon 13:30-14:30 bi-weekly (23.10, 06.11., 20.11., 04.12., 18.12., 15.01., 29.01.)
2. Location: SBC 3-E, Room 4.012
3. Appointment by email: md.tanvir.hassan@uni-hamburg.de

Definition : Let D is open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$, $x^0 \in D$. The function f is called **Partially differentiable** in x^0 with respect to x_i, if the limit

$$\frac{\partial f}{\partial x_i}(x^0) := \lim_{t \to 0} \frac{f(x^0 + te_i) - f(x^0)}{t}$$

exists. Here e_i denotes the i-th unit vector. And this limit is called partial derivative of f with respect to x_i at x^0.

If at every point x^0 the partial derivative with respect to every veriable $x_i, i = 1, ..., n$ exists and if the partial derivatives are continuous functions then we call f continuous partial differentiable or a C^1-function.

Example

Are these functions partially differentiable? Compute the partial derivatives of the given functions.

- $f_1(x, y) = 4x + 2y^2$
- $f_2(x, y) = 4x^4 - 2y^2$
- $f_3(x, y) = 4x^4 - |y|$

Compute the partial derivatives of the given function g:

$$g(x,y) = sin(2x+3) + e^{3y}$$

Solution:

$$\frac{\partial g(x,y)}{\partial x} = 2cos(2x+3) + 0$$

$$\frac{\partial g(x,y)}{\partial y} = 0 + 3e^{3y}$$

Example

Check if the function f is continuous at the origin.

$$f(x, y) = 4x + 2y^2$$

Definition : Let D is open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is partial differentiable then the gradient is

$$gradf(x_0) = \left(\frac{\partial f}{\partial x_1}(x^0), ..., \frac{\partial f}{\partial x_n}(x^0)\right)$$

The Symbolic vector

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)^T$$

and denote as a nabla-operator.
Thus we obtain the column vector

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x^0), ..., \frac{\partial f}{\partial x_n}(x^0)\right)^T$$

Example

Compute the gradient of the following functions:

- $f(x, y) = 2x^2 + 3y^2$
- $g(x, y) = 2x^2 + 3x^2y^2$
- $h(x, y) = 2cos(x) + 3e^y$

Compute the gradient of the function f:

$$f(x,y,z) = 2x^2y^4 + 3xy^2z^4$$

Solution:

$$\frac{\partial f}{\partial x} = 4xy^4 + 3y^2z^4$$

$$\frac{\partial f}{\partial y} = 8x^2y^3 + 6xyz^4$$

$$\frac{\partial f}{\partial z} = 0 + 12xy^2z^3$$

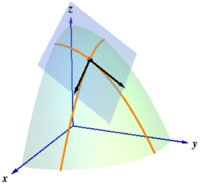
$$grad(f) = (4xy^4 + 3y^2z^4, 8x^2y^3 + 6xyz^4, 12xy^2z^3)$$

Definition : The equation of a tangent plane to a graph of a differentiable function f at the point $(x_0, y_0) \in D \subset \mathbb{R}^2$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Parameters from the function graph can be choosen

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + (x - x_0) \begin{pmatrix} 1 \\ 0 \\ f_x(x_0, y_0) \end{pmatrix} + (y - y_0) \begin{pmatrix} 0 \\ 1 \\ f_y(x_0, y_0) \end{pmatrix}.$$



Example

A differentiable function is $z = f(x, y) = 3 + 2x^2 + y^2$ and at the point $(x_0, y_0) = (3, 2)$ we want to compute the tangent plane.

Compute the tangent plane of the surface $z = x^2 + y^2$ at the point $(x_0, y_0, z_0) := (1, 2, 5)$.
Hints: $T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
Solution:

$$z_0(x_0, y_0) = x_0^2 + y_0^2 = 1^2 + 2^2 = 5 = z_0$$

$$\frac{\partial f(x_0, y_0)}{\partial x} = 2$$

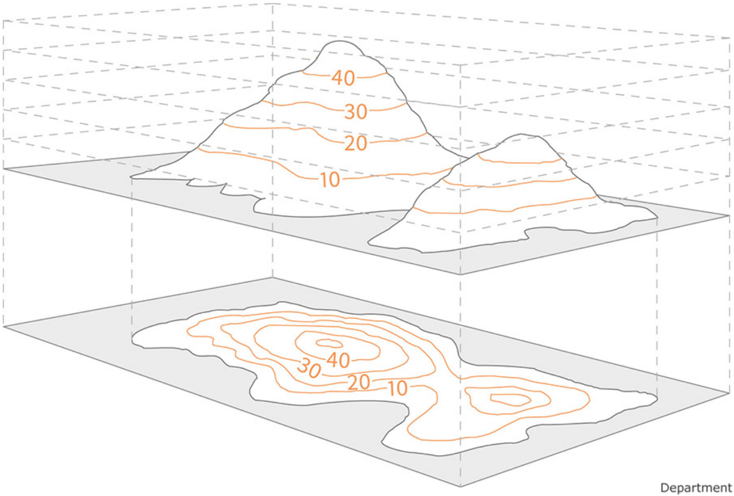
$$\frac{\partial f(x_0, y_0)}{\partial y} = 4$$

$$z = 5 + 2(x - 1) + 4(y - 2)$$

$$z = 2x + 4y - 8$$

Definition : A contour line, which is also referred to as a level curve, represents the intersection of a specific surface with a horizontal plane defined by the equation $z = c$. When these contour lines are depicted together on the xy-plane, it forms a representation known as a contour map or contour plot, offering valuable insights into the characteristics of the surface.

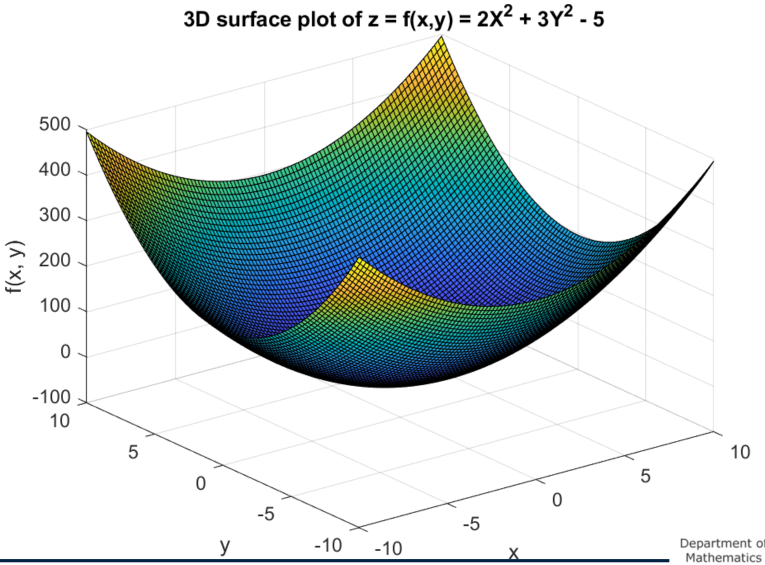
Contour maps of mountains:

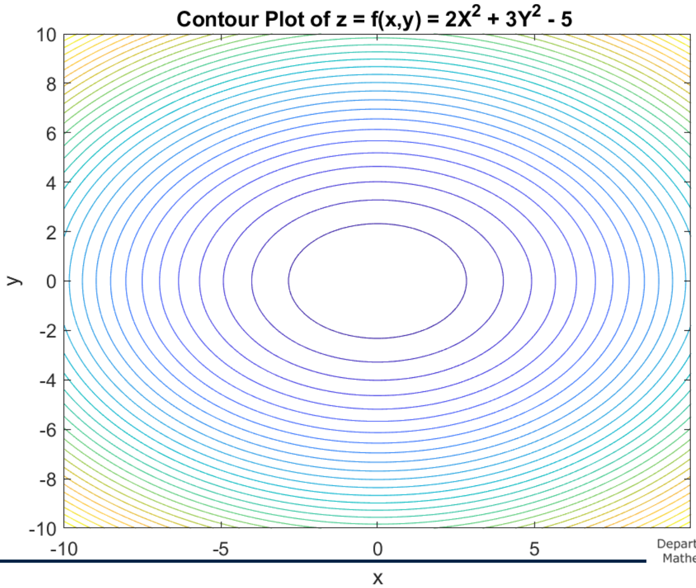


Example

Contour map for $f(x, y) = 2x^2 + 3y^2 - 5$

1. Contour maps provide valuable information about the behavior of the function in the xy plane.
2. Contour lines represent regions of equal function values.
3. $f = 5$ the center of the contour map.
4. As you move away from the center, the values of f increase.


3D surface plot of z = f(x,y) = 2X^2 + 3Y^2 - 5
f(x, y)
500
400
300
200
100
0
-100
10
5
0
-5
-10
y
-10
-5
0
5
10
x


Contour Plot of z = f(x,y) = 2X2 + 3Y2 - 5
10
8
6
4
2
0
-2
-4
-6
-8
-10
y
-10
-5
0
5
x

Plot contour lines of the given function

$$z = f(x, y) = 3 + x + y^2$$

Solution:

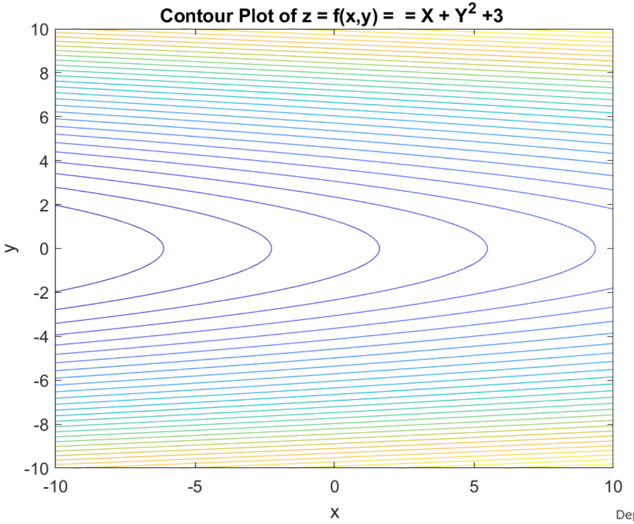
$$z = f(x, y) = 3 + x + y^2 = c$$
$$x + y^2 = c - 3$$

- $c = 3$; point
- $c > 3$; parabola
- $c < 3$; no line in the real space

Code, MATLAB


```
x = linspace(-10, 10, 100);
y = linspace(-10, 10, 100);
[X, Y] = meshgrid(x, y);
F = X + Y.^2 + 3;
contour(X, Y, F, 30);
xlabel('x');
ylabel('y');
title('name');
```

Exercise



Determine the direction of the gradient on the contour lines of the given function f and

$$z = f(x, y) = 3 + x + y^2$$

Solution:
The gradient is orthogonal to the contour lines.

Compute the second-order derivative of the function f:

$$f(x, y, z) = 2x^2y^4 + 3xy^2z^4$$

Solution:

$$f_x = 4xy^4 + 3y^2z^4$$

$$f_{xx} = 4y^4 + 0$$

$$f_y = 8x^2y^3 + 6xyz^4$$

$$f_{yy} = 24x^2y^2 + 6xz^4$$

$$f_z = 0 + 12xy^2z^3$$

$$f_{zz} = 36xy^2z^2$$

Compute the second order derivative of the function f:

$$f(x, y, z) = 2xy^2sin(z)$$

Solution:

$$f_x = 2xy^2sin(z)$$

$$f_{xx} = 0$$

$$f_y = 4xysin(z)$$

$$f_{yy} = 4xsin(z)$$

$$f_z = 2xy^2cos(z)$$

$$f_{zz} = -2xy^2sin(z)$$

Compute the partial derivatives of the given functions :

- $(ax + by)^2$
- e^{ax+by}
- $\frac{sin(ax)}{by}$

Compute the partial derivatives of the given functions :

$$f(x,y) = \frac{sin(ax)}{by}$$

Solution: Using the formula

$$\frac{\partial}{\partial x_i}\frac{f(x)}{g(x)} = \frac{\frac{\partial f(x)}{\partial x_i}.g(x) - f(x).\frac{\partial g(x)}{\partial x_i}}{g(x)^2}$$

$$\frac{\partial f}{\partial x} = \frac{(by).\frac{\partial sin(ax)}{\partial x} - sin(ax)\frac{\partial (by)}{\partial x}}{(by)^2} = \frac{acos(ax)}{by}$$

$$\frac{\partial f}{\partial y} = \frac{-bsin(ax)}{(by)^2}$$

Compute the Gradient of the given functions :

$$f(x, y) = x^4 lny - y^5 e^x$$

Solution:

$$\frac{\partial f}{\partial x} = 4x^3 lny - y^5 e^x$$

$$\frac{\partial f}{\partial y} = \frac{x^4}{y} - 5y^4 e^x$$

$$grad(f) = (4x^3 lny - y^5 e^x, \frac{x^4}{y} - 5y^4 e^x)$$

Compute the third-order derivative f_{xyy} of the given functions :

$$f(x,y) = ln(xy^2 + 2y)$$

Solution: Using the formula

$$\frac{\partial}{\partial x_i}\frac{f(x)}{g(x)} = \frac{\frac{\partial f(x)}{\partial x_i}.g(x) - f(x).\frac{\partial g(x)}{\partial x_i}}{g(x)^2}$$

$$f_x = \frac{y^2}{xy^2 + 2y} = \frac{y}{xy + 2}$$

$$f_{xy} = \frac{(xy+2) - xy}{(xy+2)^2} = \frac{2}{(xy+2)^2}$$

$$f_{xyy} = \frac{-4x}{(xy+2)^3}$$

THANK YOU