Analysis III: Auditorium Exercise-01

For Engineering Students

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Office Hours

1. Mon 13:30-14:30 bi-weekly (23.10, 06.11., 20.11., 04.12., 18.12., 15.01., 29.01.)
2. Location: SBC 3-E, Room 4.012
3. Appointment by email: md.tanvir.hassan@uni-hamburg.de

Definition : Let D is open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$, $x^0 \in D$. The function f is called **Partially differentiable** in x^0 with respect to x_i, if the limit

$$\frac{\partial f}{\partial x_i}(x^0) := \lim_{t \to 0} \frac{f(x^0 + te_i) - f(x^0)}{t}$$

exists. Here e_i denotes the i-th unit vector. And this limit is called partial derivative of f with respect to x_i at x^0.

If at every point x^0 the partial derivative with respect to every veriable $x_i, i = 1, ..., n$ exists and if the partial derivatives are continuous functions then we call f continuous partial differentiable or a C^1-function.

Example

Are these functions partially differentiable? Compute the partial derivatives of the given functions.

- $f_1(x, y) = 4x + 2y^2$
- $f_2(x, y) = 4x^4 - 2y^2$
- $f_3(x, y) = 4x^4 - |y|$

Exercise: 1

Compute the partial derivatives of the given function g:

$$g(x, y) = sin(2x + 3) + e^{3y}$$

Check if the function f is continuous at the origin.

$$f(x,y) = 4x + 2y^2$$

Definition : Let D is open and $D \subset \mathbb{R}^n$, a function $f : D \to \mathbb{R}$ is partial differentiable then the gradient is

$$gradf(x_0) = \left(\frac{\partial f}{\partial x_1}(x^0), ..., \frac{\partial f}{\partial x_n}(x^0)\right)$$

The Symbolic vector

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n}\right)^T$$

and denote as a nabla-operator.
Thus we obtain the column vector

$$\nabla f(x_0) = \left(\frac{\partial f}{\partial x_1}(x^0), ..., \frac{\partial f}{\partial x_n}(x^0)\right)^T$$

Example

Compute the gradient of the following functions:

- $f(x, y) = 2x^2 + 3y^2$
- $g(x, y) = 2x^2 + 3x^2y^2$
- $h(x, y) = 2cos(x) + 3e^y$

Exercise: 2

Compute the gradient of the function f:

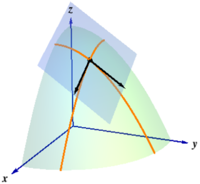
$$f(x, y, z) = 2x^2y^4 + 3xy^2z^4$$

Definition : The equation of a tangent plane to a graph of a differentiable function f at the point $(x_0, y_0) \in D \subset \mathbb{R}^2$ is

$$z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0).$$

Parameters from the function graph can be choosen

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ f(x_0, y_0) \end{pmatrix} + (x - x_0) \begin{pmatrix} 1 \\ 0 \\ f_x(x_0, y_0) \end{pmatrix} + (y - y_0) \begin{pmatrix} 0 \\ 1 \\ f_y(x_0, y_0) \end{pmatrix}.$$



Example

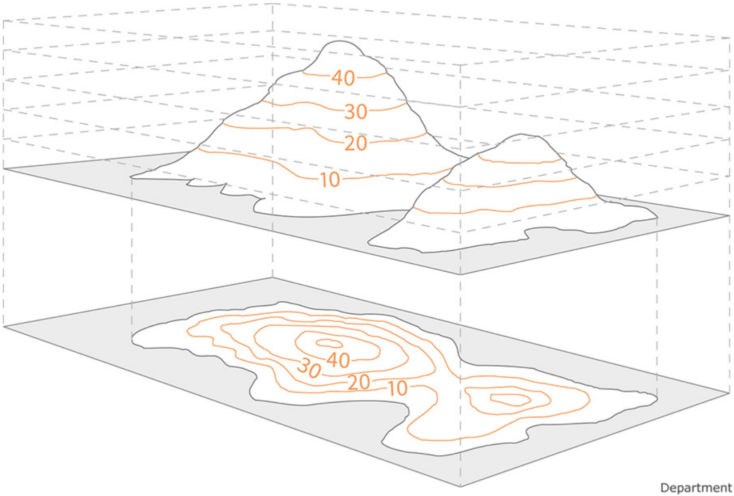
A differentiable function is $z = f(x, y) = 3 + 2x^2 + y^2$ and at the point $(x_0, y_0) = (3, 2)$ we want to compute the tangent plane.

Exercise: 3

Compute the tangent plane of the surface $z = x^2 + y^2$ at the point $(x_0, y_0, z_0) := (1, 2, 5)$.
Hints: $T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Definition : A contour line, which is also referred to as a level curve, represents the intersection of a specific surface with a horizontal plane defined by the equation $z = c$. When these contour lines are depicted together on the xy-plane, it forms a representation known as a contour map or contour plot, offering valuable insights into the characteristics of the surface.

Contour maps of mountains:

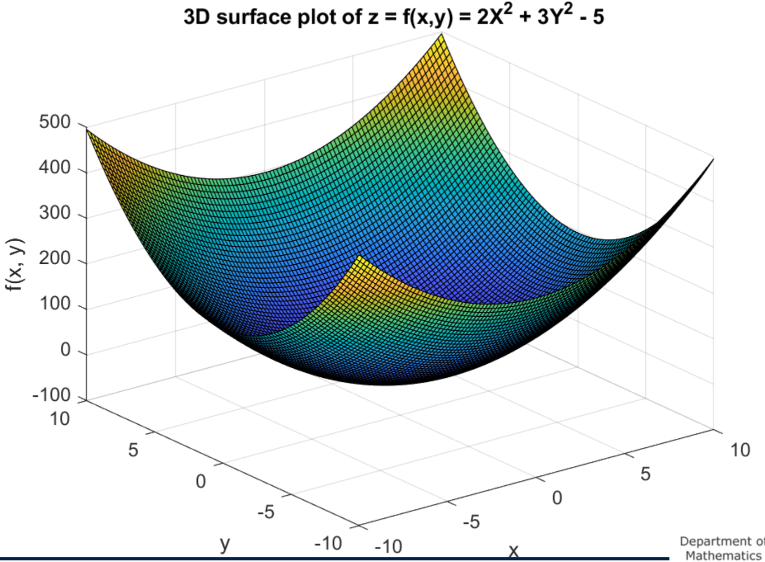


Example

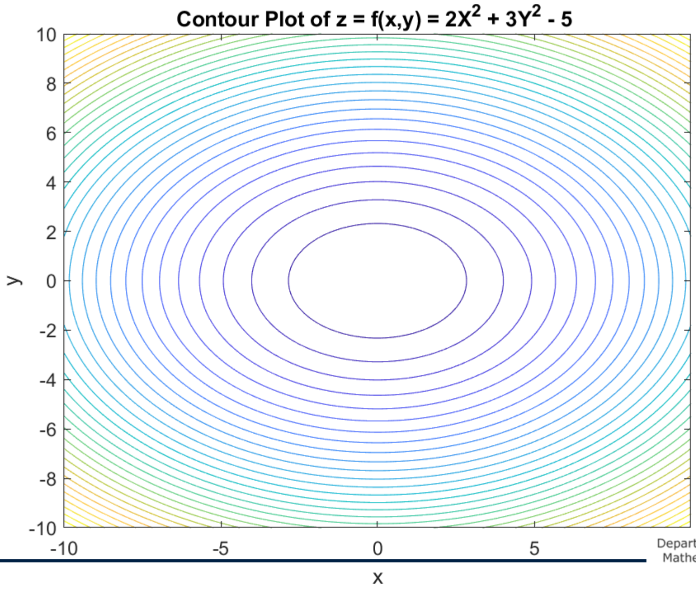
Contour map for $f(x, y) = 2x^2 + 3y^2 - 5$

1. Contour maps provide valuable information about the behavior of the function in the xy plane.
2. Contour lines represent regions of equal function values.
3. $f = 5$ the center of the contour map.
4. As you move away from the center, the values of f increase.

Example


3D surface plot of z = f(x,y) = 2X^2 + 3Y^2 - 5
f(x, y)
500
400
300
200
100
0
-100
10
5
0
-5
-10
y
-10
-5
0
5
10
x

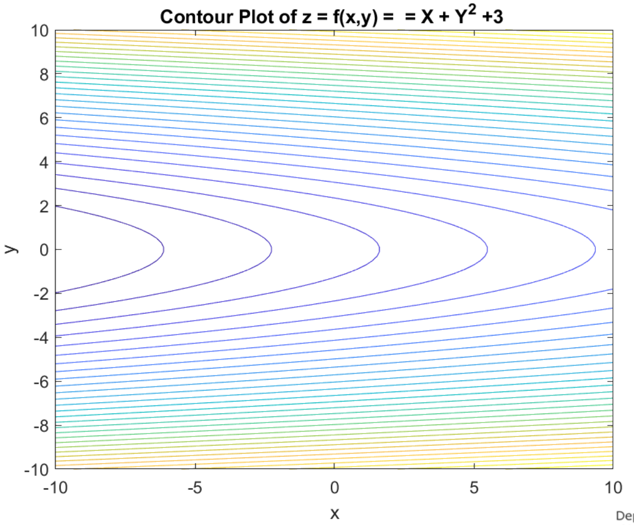
Example



Plot contour lines of the given function

$$z = f(x, y) = 3 + x + y^2$$

```
x = linspace(-10, 10, 100);
y = linspace(-10, 10, 100);
[X, Y] = meshgrid(x, y);
F = X + Y.^2 + 3;
contour(X, Y, F, 30);
xlabel('x');
ylabel('y');
title('name');
```


Contour Plot of z = f(x,y) = = X + Y2 +3
10
8
6
4
2
0
-2
-4
-6
-8
-10
y
-10
-5
0
5
10
x

Exercise: 5

Determine the direction of the gradient on the contour lines of the given function f and

$$z = f(x, y) = 3 + x + y^2$$

Compute the second order derivative of the function f:

$$f(x, y, z) = 2x^2y^4 + 3xy^2z^4$$

Compute the second order derivative of the function f:

$$f(x, y, z) = 2xy^2 sin(z)$$

Compute the partial derivatives of the given functions :

- $(ax+by)^2$
- e^{ax+by}
- $\frac{sin(ax)}{by}$

Compute the Gradient of the given functions :

$$f(x, y) = x^4 lny - y^5 e^x$$

Compute the third-order derivative f_{xyy} of the given functions :

$$f(x, y) = ln(xy^2 + 2y)$$

THANK YOU