Exercise 1: (2+2 Points)

Given a function f with

$$f(x,y) = x^2 - \cos(y)e^{x-1}.$$

a) Compute

  (i) the gradient and

  (ii) the Hessian matrix.

b) Determine the 2nd degree Taylor polynomial for the function f at the point $(x_0, y_0) = (1, 0)$.

Solution:

a) (2 points)

  (i) grad $f(x,y) = (f_x(x,y), f_y(x,y)) = (2x - \cos(y)e^{x-1}, \sin(y)e^{x-1})$

  (ii) Hess $f(x,y) = \begin{pmatrix} 2-\cos(y)e^{x-1} & \sin(y)e^{x-1} \\ \sin(y)e^{x-1} & \cos(y)e^{x-1} \end{pmatrix}$

b) (2 points)

$f(1,0) = 0,\ f_x(1,0) = 1,\ f_y(1,0) = 0$

$f_{xx}(1,0) = 1,\ f_{xy}(1,0) = 0,\ f_{yy}(1,0) = 1$

$$\begin{aligned} T_2(x,y;1,0) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)\,y \\ &\quad + \frac{1}{2}\left(f_{xx}(1,0)(x-1)^2 + 2f_{xy}(1,0)(x-1)y + f_{yy}(1,0)\,y^2\right) \\ &= x - 1 + \frac{(x-1)^2}{2} + \frac{y^2}{2} \end{aligned}$$

Alternatively:

$x^2 - \cos(y)e^{x-1}$

$= (x-1)^2 + 2(x-1) + 1 - \left(1 - \frac{y^2}{2} \pm \ldots\right)\left(1 + (x-1) + \frac{(x-1)^2}{2} \pm \ldots\right)$

$= \underbrace{x - 1 + \frac{(x-1)^2}{2} + \frac{y^2}{2}}_{=T_2(x,y;1,0)} \pm \ldots$

Exercise 2: (1+3+1 points)

Compute the extreme values of the function

$$f : \mathbb{R}^2 \to \mathbb{R} \quad \text{with} \quad f(x,y) = x^2 + y^2$$

under the constraint $g(x,y) = 9x^2 + 4y^2 - 36 = 0$ using the method of Lagrange multipliers:

a) Check the regularity condition for g.

b) Compute the candidates for extrema using the method of Lagrange multipliers.

c) Classify their type.

Solution:

a) (1 point)

Regularity condition:

$\boldsymbol{J}g(x,y) = \text{grad } g(x,y)^T = (18x, 8y) = (0,0) \Rightarrow (x,y) = (0,0)$

It holds $g(0,0) = -36$. The point $(0,0)$ belongs not to the ellipse $g = 0$. Hence all admissible points fulfill the regularity condition $\text{rank}(\boldsymbol{J}g(x,y)) = 1$.

b) (3 points)

Method of Lagrange multipliers.:

$$\begin{pmatrix} \nabla f(x,y) + \lambda \nabla g(x,y) \\ g(x,y) \end{pmatrix} = \begin{pmatrix} 2x + 18\lambda x \\ 2y + 8\lambda y \\ 9x^2 + 4y^2 - 36 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

1. equation:
1.case: $x = 0 \Rightarrow g(0,y) = 4y^2 - 36 = 0 \Rightarrow y_1 = 3,\ y_2 = -3$

Candidates for extrema: $P_1 = \begin{pmatrix} 0 \\ 3 \end{pmatrix}, \ P_2 = \begin{pmatrix} 0 \\ -3 \end{pmatrix}$

2.case:
$\lambda = -\frac{1}{9} \overset{\text{2. eq.}}{\Rightarrow} y = 0 \Rightarrow g(x,0) = 9x^2 - 36 = 0 \Rightarrow x_1 = 2\,,\ x_2 = -2$

Candidates for extrema: $P_3 = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \ P_4 = \begin{pmatrix} -2 \\ 0 \end{pmatrix}$

c) (1 point)

Since ellipse $g = 0$ is a compact set and f is continuous, function f attains on $g = 0$ its absolute maximum and minimum.

$P_{1,2}$ are global maxima and $P_{3,4}$ are global minima, since

$$f(P_{1,2}) = 9\,, \quad f(P_{3,4}) = 4\,.$$

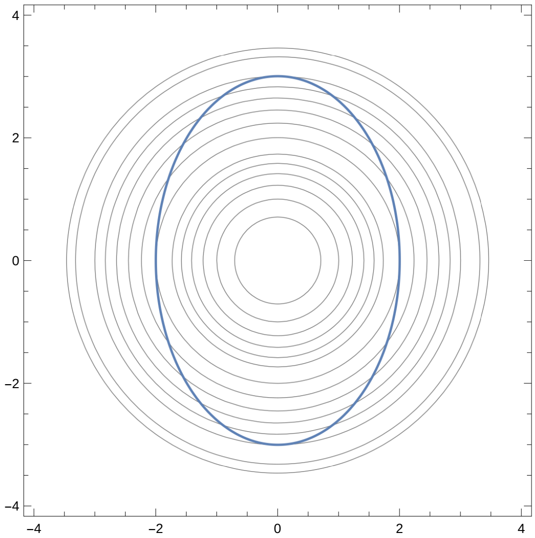

Figure 2 Constraint $g(x, y) := 9x^2 + 4y^2 - 36 = 0$ with contour lines of function $f(x, y) = x^2 + y^2$

Exercise 3: (3 points)

Compute for the vector field

$$\boldsymbol{f} : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{with} \quad \boldsymbol{f}(x,y) = \begin{pmatrix} -xy \\ y^2 \end{pmatrix}$$

the line integral $\int_{\boldsymbol{c}} \boldsymbol{f}(\boldsymbol{x})d\boldsymbol{x}$, where the curve $\boldsymbol{c}$ runs through the left half circle

$$H := \left\{(x,y)^T \in \mathbb{R}^2 \mid x^2 + y^2 = 9,\ x \leq 0\right\}$$

in a mathematically positive orientation.

Solution:

We parametrize curve $\boldsymbol{c}$ as follows $\quad \boldsymbol{c}(t) = \begin{pmatrix} 3\cos(t) \\ 3\sin(t) \end{pmatrix}, \ \frac{\pi}{2} \leq t \leq \frac{3\pi}{2}.$

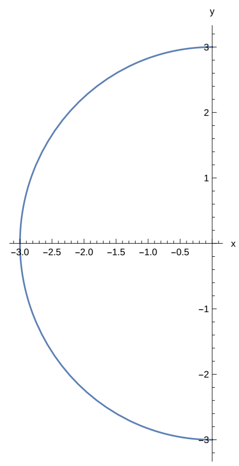

Figure 3 left half circle, radius $r = 3$

Compute the line integral with the tangent vector $\dot{\boldsymbol{c}}(t) = \begin{pmatrix} -3\sin(t) \\ 3\cos(t) \end{pmatrix}$.

$$\int_{\boldsymbol{c}} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} = \int_{\pi/2}^{3\pi/2} \langle \boldsymbol{f}(\boldsymbol{c}(t)), \dot{\boldsymbol{c}}(t)\rangle \ dt = \int_{\pi/2}^{3\pi/2} \left\langle \begin{pmatrix} -9\cos(t)\sin(t) \\ 9\sin^2(t) \end{pmatrix}, \begin{pmatrix} -3\sin(t) \\ 3\cos(t) \end{pmatrix} \right\rangle dt$$

$$= 18 \int_{\pi/2}^{3\pi/2} 3\cos(t)\sin(t)^2 \ dt \overset{subst.}{=} 18 \left(\sin^3(t)\right)\Big|_{\pi/2}^{3\pi/2} = -36$$

Exercise 4: (1+3 points)

Given a vector field $\boldsymbol{f} : \mathbb{R}^3 \to \mathbb{R}^3$ with

$$\boldsymbol{f}(x,y,z) = \begin{pmatrix} e^z \\ 4z - 2\sin(y) \\ xe^z + 4y + 1 \end{pmatrix},$$

a) Prove the existence of a potential for $\boldsymbol{f}$ without calculating it.

b) Compute a potential for $\boldsymbol{f}$.

Solution:

a) $\mathbb{R}^3$ is simply connected and the integrability condition is satisfied

$$\text{rot}\boldsymbol{f}(x,y,z) = \begin{pmatrix} f_{3y} - f_{2z} \\ f_{1z} - f_{3x} \\ f_{2x} - f_{1y} \end{pmatrix} = \begin{pmatrix} 4-4 \\ e^z - e^z \\ 0-0 \end{pmatrix} = \mathbf{0}.$$

Hence for $\boldsymbol{f}(x,y,z)$ exists a potential $v(x,y,z)$, i.e. it holds $\boldsymbol{f} = (v_x, v_y, v_z)$.

b) $v_x(x,y,z) \overset{!}{=} e^z \quad \Rightarrow \quad v(x,y,z) = xe^z + c(y,z)$

$$\begin{aligned}
&\Rightarrow \quad v_y(x,y,z) = c_y(y,z) \overset{!}{=} 4z - 2\sin(y) \\
&\Rightarrow \quad c(y,z) = 4yz + 2\cos(y) + k(z) \\
&\Rightarrow \quad v(x,y,z) = xe^z + 4yz + 2\cos(y) + k(z)
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \quad v_z(x,y,z) = xe^z + 4y + k'(z) \overset{!}{=} xe^z + 4y + 1 \Rightarrow k'(z) = 1 \Rightarrow k(z) = z + K \\
&\Rightarrow \quad v(x,y,z) = xe^z + 4yz + 2\cos(y) + z + K \quad \text{mit } K \in \mathbb{R}
\end{aligned}$$

Alternatively:

Choose as a curve $\boldsymbol{k}$ the straight line connecting points $(0,0,0)$ and (x,y,z), i.e. $\boldsymbol{k}(t) = t(x,y,z)^T$ with $0 \le t \le 1$. Now compute the potential v for $\boldsymbol{f}$ using the fundamental theorem of line integrals

$$v(x,y,z) = \int_{\boldsymbol{k}} \boldsymbol{f}(\boldsymbol{x})\, d\boldsymbol{x} + K = \int_0^1 \boldsymbol{f}(\boldsymbol{k}(t))\dot{\boldsymbol{k}}(t)\, dt + K$$

$$= \int_0^1 \left\langle \begin{pmatrix} e^{zt} \\ 4zt - 2\sin(yt) \\ xte^{zt} + 4yt + 1 \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\rangle dt + K = \int_0^1 xe^{zt} + zxte^{zt} + 8yzt - 2y\sin(yt) + z\, dt + K$$

$$= \left(xte^{zt} + 4yzt^2 + 2\cos(yt) + zt\right)\Big|_0^1 + K = xe^z + 4yz + 2\cos(y) + z + \tilde{K}.$$

Exercise 5: (2+2 points)

a) Make a sketch of the area K bounded by $0 \le y$, $0 \le z$ and $x^2 + y^2 + z^2 = 9$, and represent it using spherical coordinates.

b) Compute the mass of K with the density function $\rho = 8z + 3$ using spherical coordinates.

Solution:

a) (2 points)
$0 \le y$, $0 \le z$ and $x^2+y^2+z^2 = 9$ define a quarter sphere with the center at point $(0,0,0)$ of radius $r = 3$.

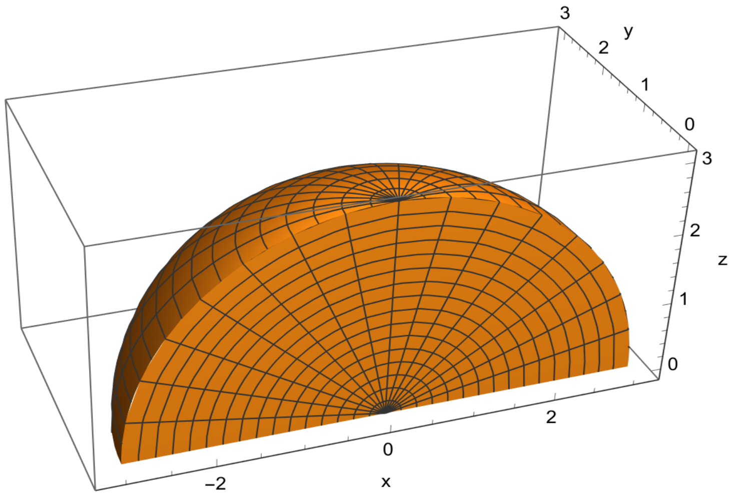

Figure 5 Quarter sphere K, radius $r = 3$

Spherical coordinates $\quad 0 \le r \le 3,\ \ 0 \le \varphi \le \pi,\ \ 0 \le \psi \le \dfrac{\pi}{2}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r,\varphi,\psi) = \begin{pmatrix} r\cos(\varphi)\cos(\psi) \\ r\sin(\varphi)\cos(\psi)) \\ r\sin(\psi) \end{pmatrix}$$

b) (2 points)
With the density $\rho = 8z + 3$ in spherical coordinates the mass of K is obtained using the transformation theorem

$$\begin{aligned} M &= \int_K 8z + 3\, d(x,y,z) = \int_0^3 \int_0^{\pi/2} \int_0^{\pi} (8r\sin(\psi) + 3)r^2\cos(\psi)\, d\varphi\, d\psi\, dr \\ &= \int_0^3 4r^3\, dr \int_0^{\pi/2} 2\sin(\psi)\cos(\psi) d\psi \int_0^{\pi} d\varphi + \int_0^3 3r^2\, dr \int_0^{\pi/2} \cos(\psi) d\psi \int_0^{\pi} d\varphi \\ &= \left(r^4\Big|_0^3\right)\left(\varphi\Big|_0^{\pi}\right)\left(\sin^2(\psi)\Big|_0^{\pi/2}\right) + \left(r^3\Big|_0^3\right)\left(\varphi\Big|_0^{\pi}\right)\left(\sin(\psi)\Big|_0^{\pi/2}\right) = (81+27)\pi = 108\pi \end{aligned}$$