Exercise 1: (5 points)
Compute all stationary points of the following function and determine their types

$$f(x,y) = x^3 - 3x + y^3 - 12y.$$

Solution:
grad $f(x,y) = (3x^2-3, 3y^2-12)^T = (3(x^2-1), 3(y^2-4))^T = (0,0)^T$
We obtain $x = 1$ or $x = -1$ and $y = 2$ or $y = -2$, so the stationary points are

$$P_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix},\ P_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},\ P_3 = \begin{pmatrix} -1 \\ 2 \end{pmatrix},\ P_4 = \begin{pmatrix} -1 \\ -2 \end{pmatrix}.$$

Hess $f(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & 6y \end{pmatrix}$

Hess $f(P_1) = \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix} \Rightarrow$ positiv definite $\Rightarrow P_1$ minimum

Hess $f(P_2) = \begin{pmatrix} 6 & 0 \\ 0 & -12 \end{pmatrix} \Rightarrow$ indefinite $\Rightarrow P_2$ saddle point

Hess $f(P_3) = \begin{pmatrix} -6 & 0 \\ 0 & 12 \end{pmatrix} \Rightarrow$ indefinite $\Rightarrow P_3$ saddle point

Hess $f(P_4) = \begin{pmatrix} -6 & 0 \\ 0 & -12 \end{pmatrix} \Rightarrow$ negativ definite $\Rightarrow P_4$ maximum

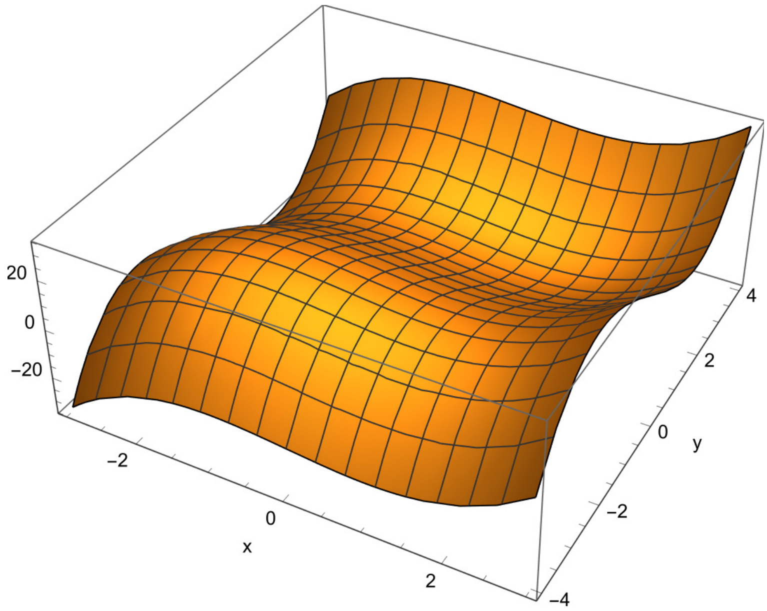

Figure 1 $f(x,y) = x^3 - 3x + y^3 - 12y$

Exercise 2: (1+1+3 points)
Given an implicit representation of a curve

$$f(x,y) := 4x^2 + 9y^2 - 36y = 0$$

a) determine the symmetries of the curve.

b) Compute the gradient of f.

c) Compute the points of curve with horizontal and vertical tangent.

Solution:

a) (1 point) The curve is symmetric about y-axis, since

$$f(-x,y) = 4(-x)^2 + 9y^2 - 36y = 4x^2 + 9y^2 - 36y = f(x,y).$$

b) (1 point)
grad $f(x,y) = (f_x(x,y), f_y(x,y)) = (8x, 18y - 36)$

c) (3 points)

(i) The points of the curve with horizontal tangent are given by the conditions

$$f_x(x,y) = 0 \quad \wedge \quad f(x,y) = 0 \quad \wedge \quad f_y(x,y) \neq 0\,.$$

$0 = f_x(x,y) = 8x \quad \Rightarrow \quad x = 0$
$\Rightarrow \quad 0 = f(0,y) = 9y^2 - 36y = 9y(y-4) \quad \Rightarrow \quad y = 0 \vee y = 4$

$$\Rightarrow \quad P_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} 0 \\ 4 \end{pmatrix}.$$

It holds $f_y(0,0) = -36 \neq 0$ and $f_y(0,4) = 36 \neq 0$.

(ii) The points of the curve with vertical tangent are given by the conditions

$$f_y(x,y) = 0 \quad \wedge \quad f(x,y) = 0 \quad \wedge \quad f_x(x,y) \neq 0\,.$$

$0 = f_y(x,y) = 18y - 36 \quad \Rightarrow \quad y = 2$
$\Rightarrow \quad 0 = f(x,2) = 4x^2 + 9 \cdot 2^2 - 72 = 4x^2 - 36 \quad \Rightarrow \quad x_1 = 3 \vee x_2 = -3$

$$\Rightarrow \quad P_3 = \begin{pmatrix} 3 \\ 2 \end{pmatrix}, \; P_4 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}.$$

It holds $f_x(\pm 3, 2) = \pm 24 \neq 0$.

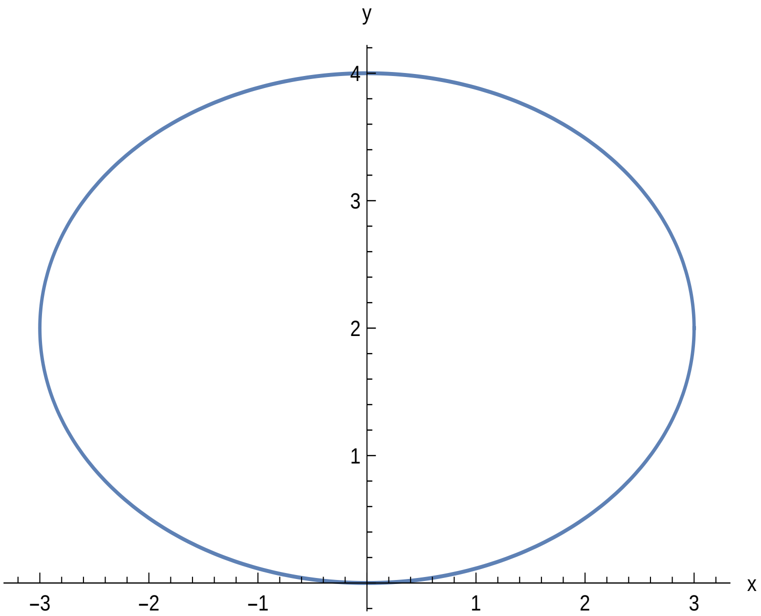

Figure 2 Ellipse $\dfrac{x^2}{3^2} + \dfrac{(y-2)^2}{2^2} = 1 \Leftrightarrow 4x^2 + 9y^2 - 36y = 0.$

Exercise 3: (2+2 points)

a) Make a sketch of the area Z enclosed by $0 \leq z \leq 5$ and $x^2 + y^2 \leq 4$, and give its representation in cylindrical coordinates.

b) Given density $\rho(x, y, z) = 2z + 1$ compute the moment of inertia of Z about z-axis using cylndrical coordinates.

Solution:

a) (2 points)

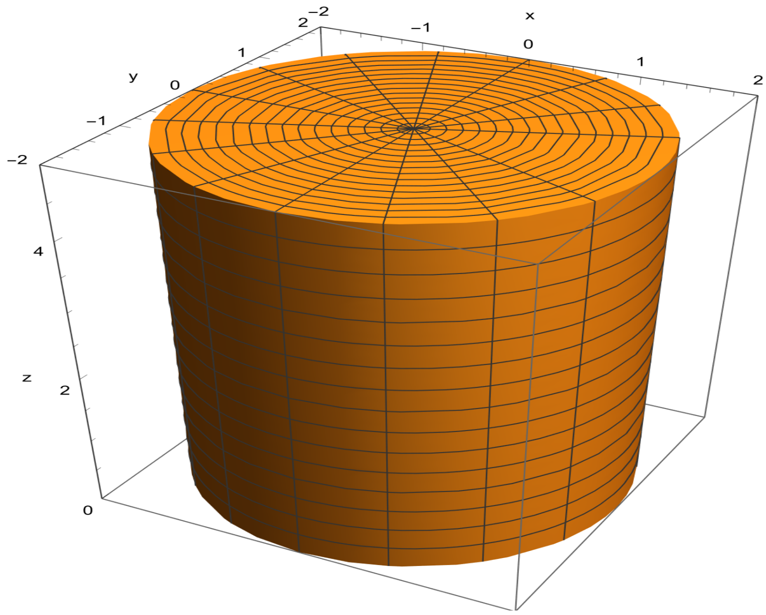

Figure 3 Cylinder Z, radius $r = 2$, height $h = 5$

Cylindrical coordinates with $0 \leq r \leq 2,\ 0 \leq \varphi \leq 2\pi, 0 \leq z \leq 5$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, \psi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

b) (2 points)

Calculation of the moment of inertia via the transformation theorem

$$\begin{aligned}
\Theta_{z\text{-axis}} &= \int_Z \rho(x, y, z) r^2(x, y, z)\, d(x, y, z) = \int_Z (2z + 1)(x^2 + y^2)\, d(x, y, z) \\
&= \int_0^2 \int_0^{2\pi} \int_0^5 (2z + 1) \cdot r^2 \cdot r\, dz\, d\varphi\, dr = \int_0^2 r^3 dr \int_0^{2\pi} d\varphi \int_0^5 2z + 1\, dz \\
&= \left(\frac{r^4}{4}\Big|_0^2 \right) \left(\varphi\Big|_0^{2\pi} \right) \left(z^2 + z\Big|_0^5 \right) = \frac{2^4}{4} \cdot 2\pi \cdot 30 = 240\pi
\end{aligned}$$

Exercise 4: (1+1+3+1 points)
Given a vector field $\boldsymbol{f}(x,y,z) = (0,0,z)^T$ and a body

$$K = \left\{ (x,y,z)^T \in \mathbb{R}^3 \,\middle|\, x^2+y^2+z^2 \leq 25 \right\},$$

a) make a sketch of K.

b) Give a parameterization for the surface S of the body K .

c) Calculate the flow(flux) of $\boldsymbol{f}$ through the surface S using parameterization from b).

d) Compute the volume integral $\displaystyle\int_K \operatorname{div} \boldsymbol{f}(x,y,z)\, d(x,y,z)$.

Solution:

a) (1 point)

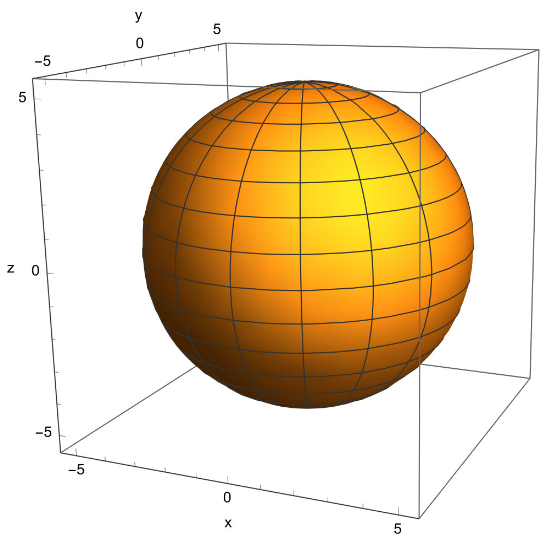

Figure 4 Sphere K, Radius $r = 5$

b) (1 point)

Parameterization of the surface of Sphere H: $\boldsymbol{q} : [0, 2\pi] \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to \mathbb{R}^3$ with

$$\boldsymbol{q}(\varphi,\psi) = \begin{pmatrix} 5\cos(\varphi)\cos(\psi) \\ 5\sin(\varphi)\cos(\psi) \\ 5\sin(\psi) \end{pmatrix}$$

c) (3 points)

Flux through H, with the outward going normal vectors

$$\frac{\partial \boldsymbol{q}}{\partial \varphi} \times \frac{\partial \boldsymbol{q}}{\partial \psi} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -5\sin(\varphi)\cos(\psi) & 5\cos(\varphi)\cos(\psi) & 0 \\ -5\cos(\varphi)\sin(\psi) & -5\sin(\varphi)\sin(\psi) & 5\cos(\psi) \end{vmatrix} = 25\cos(\psi) \begin{pmatrix} \cos(\varphi)\cos(\psi) \\ \sin(\varphi)\cos(\psi) \\ \sin(\psi) \end{pmatrix}$$

$$\int_S \boldsymbol{f}\, do \;=\; \int\limits_0^{2\pi}\int\limits_{-\pi/2}^{\pi/2} 25\cos(\psi) \left\langle \begin{pmatrix} 0 \\ 0 \\ 5\sin(\psi) \end{pmatrix}, \begin{pmatrix} \cos(\varphi)\cos(\psi) \\ \sin(\varphi)\cos(\psi) \\ \sin(\psi) \end{pmatrix} \right\rangle d\psi d\varphi$$

$$= 125 \int\limits_0^{2\pi}\int\limits_{-\pi/2}^{\pi/2} \sin^2(\psi)\cos(\psi) d\psi d\varphi = 125 \left(\varphi\big|_0^{2\pi}\right) \left(\frac{\sin^3(\psi)}{3}\bigg|_{-\pi/2}^{\pi/2} \right) = \frac{500\pi}{3}$$

d) (1 point)

With the Gauss's theorem (Divergence theorem) we get:

$$\int_K \operatorname{div} \boldsymbol{f}\, d(x,y,z) = \int_S \boldsymbol{f}\, do = \frac{500\pi}{3}$$

Alternatively:
direct calculation using spherical coordinates and $\operatorname{div} \boldsymbol{f}(x,y,z) = 1$

$$\begin{aligned} & \int_K \operatorname{div} \boldsymbol{f}(x,y,z)\, d(x,y,z) = \int_K d(x,y,z) \\ = & \int\limits_0^5 \int\limits_0^{2\pi} \int\limits_{-\pi/2}^{\pi/2} r^2 \cos(\psi)\, d\psi d\varphi dr = \int\limits_0^5 r^2 dr \int\limits_0^{2\pi} d\varphi \int\limits_{-\pi/2}^{\pi/2} \cos(\psi)\, d\psi \\ = & \left.\frac{r^3}{3}\right|_0^5 \cdot \varphi\big|_0^{2\pi} \cdot \sin(\psi)\big|_{-\pi/2}^{\pi/2} = \frac{500\pi}{3} \end{aligned}$$

Alternatively:
direct calculation using $\operatorname{div} \boldsymbol{f}(x,y,z) = 1$ and sphere volume $V = \frac{4\pi r^3}{3}$

$$\int_K \operatorname{div} \boldsymbol{f}(x,y,z)\, d(x,y,z) = \int_K d(x,y,z) = \frac{4\pi 5^3}{3} = \frac{500\pi}{3}$$