Exercise 1: (4 points)

Compute all stationary points of the following function and determine their types

$$f(x,y) = \frac{x^4}{4} - 2x^2 + \frac{y^2}{2} + 2y.$$

Solution:

grad $f(x,y) = (x^3 - 4x, y+2)^T = (x(x^2-4), y+2))^T = (0,0)^T$

We obtain $x_1 = 0,\ x_2 = 2$ or $x_3 = -2$ and $y = -2$, so the stationary points are

$$P_1 = \begin{pmatrix} 0 \\ -2 \end{pmatrix},\ P_2 = \begin{pmatrix} 2 \\ -2 \end{pmatrix},\ P_3 = \begin{pmatrix} -2 \\ -2 \end{pmatrix}.$$

Hess $f(x,y) = \begin{pmatrix} 3x^2-4 & 0 \\ 0 & 1 \end{pmatrix}$

Hess $f(P_1) = \begin{pmatrix} -4 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ eigenvalues $\lambda_1 = -4, \lambda_1 = 1 \Rightarrow$ indefinite $\Rightarrow P_1$ saddle point

Hess $f(P_2) = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ eigenvalues $\lambda_1 = 8, \lambda_1 = 1 \Rightarrow$ positive definite $\Rightarrow P_2$ minimum

Hess $f(P_3) = \begin{pmatrix} 8 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow$ eigenvalues $\lambda_1 = 8, \lambda_1 = 1 \Rightarrow$ positive definite $\Rightarrow P_3$ minimum

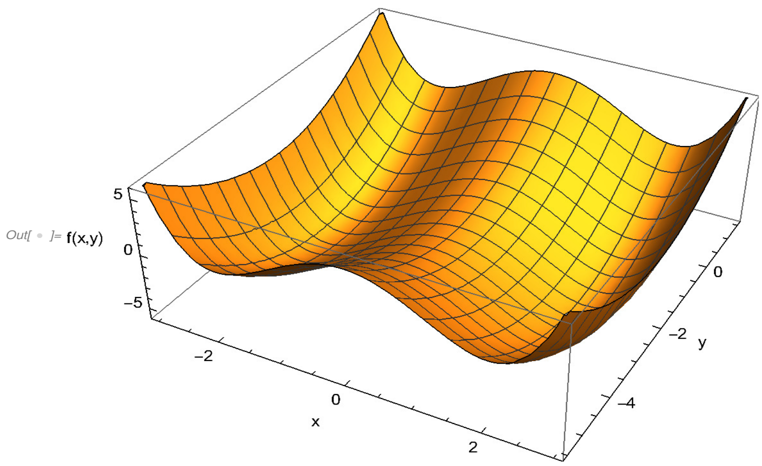

Figure 1 $f(x,y) = \dfrac{x^4}{4} - 2x^2 + \dfrac{y^2}{2} + 2y$

Exercise 2: (1+1+3 points)

Given an implicit representation of a curve

$$f(x,y) := x^2 - 6x + 4y^2 + 5 = 0,$$

a) check the symmetries of the curve.

b) Compute the gradient of f.

c) Compute the points of curve with horizontal and vertical tangent.

Solution:

a) (1 point) The curve is symmetric about x-axis, since

$$f(x,-y) = x^2 - 6x + 4(-y)^2 + 5 = f(x,y).$$

b) (1 point)
grad $f(x,y) = (f_x(x,y), f_y(x,y)) = (2x-6, 8y)$

c) (3 points)

(i) The points of the curve with horizontal tangent are given by the conditions

$$f_x(x,y) = 0 \quad \wedge \quad f(x,y) = 0 \quad \wedge \quad f_y(x,y) \neq 0\,.$$

$0 = f_x(x,y) = 2x - 6 \quad \Rightarrow \quad x = 3$
$\Rightarrow \quad 0 = f(3,y) = 3^2 - 18 + 4y^2 + 5 \quad \Rightarrow \quad y^2 = 1 \vee y_{1,2} = \pm 1$

$$\Rightarrow \quad P_1 = \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \; P_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}.$$

It holds $f_y(3, \pm 1) \neq 0$.

(ii) The points of the curve with vertical tangent are given by the conditions

$$f_y(x,y) = 0 \quad \wedge \quad f(x,y) = 0 \quad \wedge \quad f_x(x,y) \neq 0\,.$$

$0 = f_y(x,y) = 8y \quad \Rightarrow \quad y = 0$
$\Rightarrow \quad 0 = f(x,0) = x^2 - 6x + 5 = (x-1)(x-5) \quad \Rightarrow \quad x_1 = 1 \vee x_2 = 5$

$$\Rightarrow \quad P_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; P_4 = \begin{pmatrix} 5 \\ 0 \end{pmatrix}.$$

It holds $f_x(x_{1,2}, 0) \neq 0$.

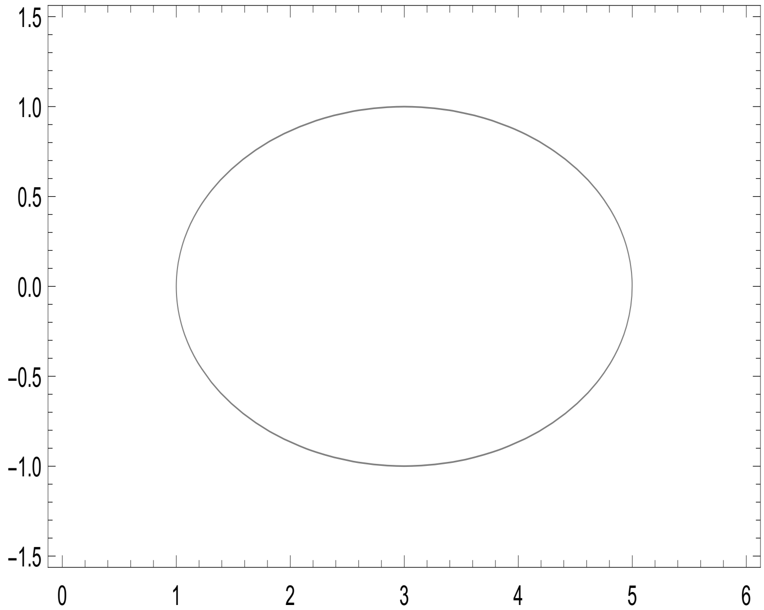

Figure 2 Ellipse $\dfrac{(x-3)^2}{2^2} + y^2 = 1 \Leftrightarrow x^2 - 6x + 4y^2 + 5 = 0.$

Exercise 3: (2+2 points)

a) Make a sketch of the area Z enclosed by $1 \le z \le 2$ and $x^2 + y^2 \le 9$, and give its representation in cylindrical coordinates.

b) Given density $\rho(x, y, z) = z^2$ compute the moment of inertia of Z about z-axis using cylindrical coordinates.

Solution:

a)

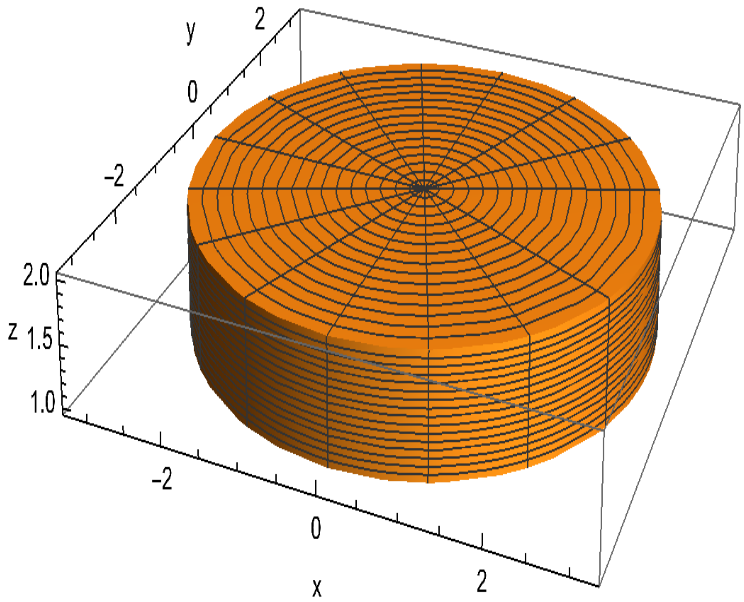

Figure 3 Cylinder Z

Cylindrical coordinates with $0 \le r \le 3,\ 0 \le \varphi \le 2\pi, 1 \le z \le 2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \Phi(r, \varphi, \psi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ z \end{pmatrix}$$

b) Calculation of the moment of inertia via the transformation theorem

$$\begin{aligned}
\Theta_{z\text{-axis}} &= \int\limits_Z \rho(x, y, z) r^2(x, y, z)\, d(x, y, z) = \int\limits_Z z^2(x^2 + y^2)\, d(x, y, z) \\
&= \int\limits_0^3 \int\limits_0^{2\pi} \int\limits_1^2 z^2 \cdot r^2 \cdot r\, dz\, d\varphi\, dr \ = \int\limits_0^3 r^3 dr \int\limits_0^{2\pi} d\varphi \int\limits_1^2 z^2\, dz \\
&= \left(\frac{r^4}{4}\bigg|_0^3 \right) \left(\varphi\big|_0^{2\pi} \right) \left(\frac{z^3}{3}\bigg|_1^2 \right) \ = \frac{3^4}{4} \cdot 2\pi \cdot \frac{7}{3} = \frac{189\pi}{2}
\end{aligned}$$

Exercise 4: (1+2+3+1 points)

Given a vector field $\boldsymbol{f}(x,y,z) = (0, yz, 0)^T$ and a body

$$K = \left\{ (x,y,z)^T \in \mathbb{R}^3 \;\middle|\; x^2 + y^2 + z^2 \leq 4\,,\; z \leq 0 \right\}\,,$$

a) make a sketch of K.

b) Give parameterizations for each of the surface segments bounding K.

c) Calculate the flow(flux) of $\boldsymbol{f}$ through these boundary segments.

Hint: It holds $\displaystyle\int \sin^2(\varphi) d\varphi = \frac{1}{2}(\varphi - \sin(\varphi)\cos(\varphi))$.

d) Compute the volume integral $\displaystyle\int_K \operatorname{div} \boldsymbol{f}(x,y,z)\, d(x,y,z)\,.$

Solution:

a) (1 point)

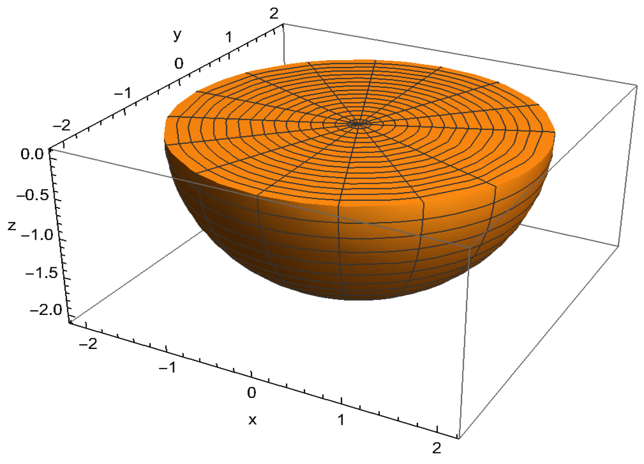

Figure 4 Hemisphere K

b) (2 points)

Parameterization of the circle face S: $\boldsymbol{p} : [0,2] \times [0, 2\pi] \to \mathbb{R}^3$ mit

$$\boldsymbol{p}(r,\varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ 0 \end{pmatrix}$$

Parameterization of the surface of hemisphere H: $\boldsymbol{q} : [0, 2\pi] \times \left[-\frac{\pi}{2}, 0\right] \to \mathbb{R}^3$ mit

$$\boldsymbol{q}(\varphi,\psi) = \begin{pmatrix} 2\cos(\varphi)\cos(\psi) \\ 2\sin(\varphi)\cos(\psi) \\ 2\sin(\psi) \end{pmatrix}$$

c) (3 points)

Flux through S, with the outward going normal vectors

$$\frac{\partial \boldsymbol{p}}{\partial r} \times \frac{\partial \boldsymbol{p}}{\partial \varphi} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \cos(\varphi) & \sin(\varphi) & 0 \\ -r\sin(\varphi) & r\cos(\varphi) & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

$$\int_S \boldsymbol{f}\, do = \int_0^2 \int_0^{2\pi} \left\langle \begin{pmatrix} 0 \\ r\sin(\varphi)\cdot 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix} \right\rangle d\varphi dr = \int_0^2 \int_0^{2\pi} 0\, d\varphi dr = 0$$

Flux through H, with the outward going normal vectors

$$\frac{\partial \boldsymbol{q}}{\partial \varphi} \times \frac{\partial \boldsymbol{q}}{\partial \psi} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ -2\sin(\varphi)\cos(\psi) & 2\cos(\varphi)\cos(\psi) & 0 \\ -2\cos(\varphi)\sin(\psi) & -2\sin(\varphi)\sin(\psi) & 2\cos(\psi) \end{vmatrix} = 4\cos(\psi) \begin{pmatrix} \cos(\varphi)\cos(\psi) \\ \sin(\varphi)\cos(\psi) \\ \sin(\psi) \end{pmatrix}$$

$$\int_T \boldsymbol{f}\, do = \int_0^{2\pi} \int_{-\pi/2}^{0} 4\cos(\psi) \left\langle \begin{pmatrix} 0 \\ 2\sin(\varphi)\cos(\psi) 2\sin(\psi) \\ 0 \end{pmatrix}, \begin{pmatrix} \cos(\varphi)\cos(\psi) \\ \sin(\varphi)\cos(\psi) \\ \sin(\psi) \end{pmatrix} \right\rangle d\psi d\varphi$$

$$= 16 \int_0^{2\pi} \int_{-\pi/2}^{0} \sin^2(\varphi)\sin(\psi)\cos^3(\psi) d\psi d\varphi = 16 \int_0^{2\pi} \int_{-\pi/2}^{0} \sin^2(\varphi) \left(-\cos^3(\psi)\right) d(\cos(\psi)) d\varphi$$

$$= -16 \left(\frac{1}{2}(\varphi - \sin(\varphi)\cos(\varphi)) \Big|_0^{2\pi} \right) \left(\frac{\cos^4(\psi)}{4} \Big|_{-\pi/2}^{0} \right) = -4\pi$$

d) (1 point)

With the Gauss's theorem (Divergence theorem) we get:

$$\int_K \operatorname{div} \boldsymbol{f}\, d(x,y,z) = \int_S \boldsymbol{f}\, do + \int_H \boldsymbol{f}\, do = -4\pi$$

Alternatively: direct calculation using spherical coordinates $\operatorname{div} \boldsymbol{f}(x,y,z) = z$

$$\int_K \operatorname{div} \boldsymbol{f}(x,y,z)\, d(x,y,z) = \int_K z\, d(x,y,z)$$

$$= \int_0^2 \int_0^{2\pi} \int_{-\pi/2}^{0} r\sin(\psi) r^2 \cos(\psi)\, d\psi d\varphi dr$$

$$= \int_0^2 r^3 dr \int_0^{2\pi} d\varphi \int_{-\pi/2}^{0} \cos(\psi)\sin(\psi)\, d\psi = \frac{r^4}{4}\Big|_0^2 \cdot \varphi\big|_0^{2\pi} \cdot \frac{\sin^2(\psi)}{2}\Big|_{-\pi/2}^{0} = -4\pi$$