

# Analysis III

## for Engineering Students

### Work sheet 7

**Exercise 1:**

a) Let  $\mathbf{f}$  be the vector field  $\mathbf{f}(x, y) = \begin{pmatrix} x^2 \\ y^2 \end{pmatrix}$ ,  $\mathbf{c}_1$  be the curve with the parametrization

$$\mathbf{c}_1(t) = (t, \sin(t)) \quad t \in [0, \pi]$$

and  $c_2$  be the mathematically positive-oriented edge of the rectangle

$$R = \{(x, y) : x \in [0, 1], y \in [0, 2]\} = [0, 1] \times [0, 2].$$

- (i) Does vector field  $f$  have potential?
- (ii) For  $i = 1, 2$  compute the line integrals

$$\int_{\mathbf{c}_i} \mathbf{f}(x, y) d(x, y).$$

- (iii) Compute the flow of  $\mathbf{f}$  through  $R$ .

b) Let  $\tilde{\mathbf{f}}$  be the vector field  $\tilde{\mathbf{f}}(x, y) = \begin{pmatrix} x^2 - y^3 \\ y^2 + x^3 \end{pmatrix}$ ,  $\mathbf{c}_2$  be defined as above and

$$\mathbf{c}_3(t) = (1, t^2) \quad t \in [0, 3].$$

Compute the line integrals

$$\int_{\mathbf{c}_2} \tilde{\mathbf{f}}(x, y) d(x, y), \quad \int_{\mathbf{c}_3} \tilde{\mathbf{f}}(x, y) d(x, y).$$

**Solution**

a) (i)  $\operatorname{rot} \mathbf{f} = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = 0$ . Potential:  $\phi = \frac{1}{3}(x^3 + y^3)$ .

(ii)  $\int_{c_1} \mathbf{f} \cdot d(x, y) = \phi(c_1(\pi)) - \phi(c_1(0)) = \phi\begin{pmatrix} \pi \\ 0 \end{pmatrix} - \phi\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \frac{\pi^3}{3}$

Since  $c_2$  is closed, it holds  $\oint_{c_2} \mathbf{f} \cdot d(x, y) = 0$

(iii) For the flow through  $R$  it holds

$$\begin{aligned} F &= \int_0^1 \int_0^2 \operatorname{div} \mathbf{f}(x, y) dy dx = \int_0^1 \int_0^2 2x + 2y dy dx = \int_0^1 [2xy + y^2]_0^2 dx \\ &= \int_0^1 4x + 4 dx = 6. \end{aligned}$$

b)  $\operatorname{rot} \tilde{\mathbf{f}} = \frac{\partial \tilde{f}_2}{\partial x} - \frac{\partial \tilde{f}_1}{\partial y} = 3x^2 + 3y^2$ .

For the direct calculation of the line integral of  $\tilde{\mathbf{f}}$  over  $\mathbf{c}_2$  one would have to parametrize the edges and compute line integrals over the individual edges. It's easier using the Green's theorem:

$$\begin{aligned} \int_{\partial R} \tilde{\mathbf{f}}(x, y) d(x, y) &= \int_R \operatorname{rot} \tilde{\mathbf{f}}(x, y) d(x, y) = \int_0^1 \int_0^2 3x^2 + 3y^2 dy dx \\ &= \int_0^1 [3x^2 y + y^3]_0^2 dx = \int_0^1 [6x^2 + 8] dx = [2x^3 + 8x]_0^1 = 10. \end{aligned}$$

For  $\mathbf{c}_3$  we compute:

$$\mathbf{c}_3(t) = \begin{pmatrix} 1 \\ t^2 \end{pmatrix} \quad \dot{\mathbf{c}}_3(t) = \begin{pmatrix} 0 \\ 2t \end{pmatrix} \quad \tilde{\mathbf{f}}(\mathbf{c}_3(t)) = \begin{pmatrix} \cdots \\ t^4 + 1 \end{pmatrix}$$

$$\int_{\mathbf{c}_3} \tilde{\mathbf{f}}(x, y) d(x, y) = \int_0^3 \langle \tilde{\mathbf{f}}(\mathbf{c}_3(t)), \dot{\mathbf{c}}_3(t) \rangle dt = \int_0^3 2t(t^4 + 1) dt = \left[ \frac{t^6}{3} + t^2 \right]_0^3 = 3^5 + 3^2 = 252.$$

**Exercise 2) ( 5 + 1 + 3 + 1 Points)**

Given are

$$K := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : 0 \leq x^2 + y^2 + z^2 \leq 16, y \geq 0 \right\},$$

and the vector field

$$\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \mathbf{f}(x, y, z) = \begin{pmatrix} x + y^2 \\ 2y \\ 3z + x^2 \end{pmatrix}.$$

a) Compute the  $\int_K \operatorname{div} \mathbf{f}(x, y, z) d(x, y, z).$

b)  $K$  is bounded by a flat surface  $W$  and a curved surface  $M$ . Provide the parametrization of  $W$ .

c) Compute the flow of  $\mathbf{f}$  through  $W$ , i.e

$$\int_W \mathbf{f} \cdot dO.$$

d) According to a) and c), how large is the flow through the curved part of the edges of  $K$ , i.e

$$\int_M \mathbf{f} \cdot dO ?$$

**Solution:**

a)  $\operatorname{div} \mathbf{f}(x, y, z) = 1 + 2 + 3 = 6.$

Parametrization of  $K$ :

Spherical coordinates:  $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos(\varphi) \cos(\theta) \\ r \sin(\varphi) \cos(\theta) \\ r \sin(\theta) \end{pmatrix}$

$$0 \leq r^2 = x^2 + y^2 + z^2 \leq 16 \implies r \in [0, 4], \quad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$$

$$y = r \sin(\varphi) \cos(\theta) \geq 0 \implies \varphi \in [0, \pi]$$

$$\begin{aligned} \int_K \operatorname{div} \mathbf{f}(x, y, z) d(x, y, z) &= \int_0^4 \int_0^\pi \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 6 \cdot r^2 \cos(\theta) d\theta d\phi dr \\ &= \int_0^4 \int_0^\pi 6r^2 [\sin(\theta)]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\phi dr \\ &= \int_0^4 12r^2 [\varphi]_0^\pi dr = 4\pi \int_0^4 3r^2 dr = 4\pi(4^3 - 0^3). \end{aligned}$$

b) Parametrization of  $W$ : a disk with radius 4 centered at 0 and  $y = 0$ .

$$p(r, \theta) = (r \cos(\theta), 0, r \sin(\theta))^T, \quad \theta \in [0, 2\pi], \quad 0 \leq r \leq 4.$$

$$\text{c) } \frac{\partial p}{\partial r} = \begin{pmatrix} \cos(\theta) \\ 0 \\ \sin(\theta) \end{pmatrix}, \quad \frac{\partial p}{\partial \theta} = \begin{pmatrix} -r \sin(\theta) \\ 0 \\ r \cos(\theta) \end{pmatrix}$$

$$\frac{\partial p}{\partial \theta} \times \frac{\partial p}{\partial r} = \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}.$$

$$< \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}, \mathbf{f}(p(r, \theta)) > = < \begin{pmatrix} 0 \\ -r \\ 0 \end{pmatrix}, \begin{pmatrix} \dots \\ 0 \\ \dots \end{pmatrix} > = 0.$$

Also  $\int_W \mathbf{f} \cdot dO = 0$ .

d) Following Gauss theorem and using a) and b) we have

$$\int_M \mathbf{f} \cdot dO = \int_K \operatorname{div} \mathbf{f}(x, y, z) d(x, y, z) - \int_W \mathbf{f} \cdot dO = 4^4 \pi.$$

**Discussion:** 24.01–26.01.22