

# Analysis III

## for Engineering Students

### Homework sheet 7

**Exercise 1:**

Given vector fields  $\mathbf{f}, \mathbf{g} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,

$$\mathbf{f} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2xz \\ -2yz \\ x^2 - y^2 \end{pmatrix} \quad \text{und} \quad \mathbf{g} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x^2 + z \\ y^2z + z^3 \\ -y \end{pmatrix}$$

a) Compute the potentials of  $\mathbf{f}$  and  $\mathbf{g}$ , if it is possible.

b) Given

$$\mathbf{c} : [0, \frac{\pi}{6}] \rightarrow \mathbb{R}^3, \quad \mathbf{c}(t) = \begin{pmatrix} t \\ \cos(3t) \\ \sin(3t) \end{pmatrix}.$$

Compute the line integrals

$$\int_{\mathbf{c}} \mathbf{f} d\mathbf{x}, \quad \text{and} \quad \int_{\mathbf{c}} \mathbf{g} d\mathbf{x}.$$

**Solution sketch:**

a)

Potential of  $\mathbf{f}$  : [3 Points]

$$\Phi_x = 2xz \iff \Phi(x, y, z) = x^2z + C(y, z)$$

$$\begin{aligned} \Phi_y &= C_y(y, z) = -2yz \iff C(y, z) = -y^2z + d(z) \iff \Phi(x, y, z) \\ &= (x^2 - y^2)z + d(z) \end{aligned}$$

$$\Phi_z = x^2 - y^2 + d'(z) = x^2 - y^2 \iff d(z) = k \iff \Phi(x, y, z) = (x^2 - y^2)z + k.$$

For example, since  $(g_1)_z = 1 \neq 0 = (g_3)_x$  one obtains, that there is no potential for  $\mathbf{g}$ . [1 Point]

b)

$$\int_{\mathbf{c}} \mathbf{f} d\mathbf{x} = \Phi(c(\pi/6)) - \Phi(c(0)) = \Phi \begin{pmatrix} \frac{\pi}{6} \\ \cos(\frac{\pi}{2}) \\ \sin(\frac{\pi}{2}) \end{pmatrix} - \Phi \begin{pmatrix} 0 \\ \cos(0) \\ \sin(0) \end{pmatrix} \quad [1 \text{ Point}]$$

$$= \Phi \begin{pmatrix} \frac{\pi}{6} \\ 0 \\ 1 \end{pmatrix} - \Phi \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \frac{\pi^2}{36} \quad [1 \text{ Point}]$$

$$\int_{\mathbf{c}} \mathbf{g} d\mathbf{x} = \int_0^{\pi/6} \langle \mathbf{g}(c(t)), \dot{\mathbf{c}}(t) \rangle dt \quad \dot{\mathbf{c}}(t) = \begin{pmatrix} 1 \\ -3 \sin(3t) \\ 3 \cos(3t) \end{pmatrix} \quad [1 \text{ Point}]$$

$$\mathbf{g}(\mathbf{c}(t)) = \begin{pmatrix} t^2 + \sin(3t) \\ \cos^2(3t) \sin(3t) + \sin^3(3t) \\ -\cos(3t) \end{pmatrix} = \begin{pmatrix} t^2 + \sin(3t) \\ \sin(3t) \\ -\cos(3t) \end{pmatrix} \quad [1 \text{ Point}]$$

$$\mathbf{g}(\mathbf{c}(t))^T \cdot \dot{\mathbf{c}}(t) = t^2 + \sin(3t) - 3 \sin^2(3t) - 3 \cos^2(3t) = t^2 + \sin(3t) - 3 \quad [1 \text{ Point}]$$

$$\int_{\mathbf{c}} \mathbf{g} d\mathbf{x} = \int_0^{\pi/6} (t^2 + \sin(3t) - 3) dt = \left[ \frac{t^3}{3} - \frac{\cos(3t)}{3} - 3t \right]_0^{\pi/6}$$

$$= \frac{\pi^3}{3 \cdot 6^3} - \frac{\pi}{2} + \frac{1}{3}. \quad [1 \text{ Point}]$$

**Exercise 2:**

Given the body  $K := \{ \mathbf{x} \in \mathbb{R}^3 \mid x^2 + y^2 \leq 4, 0 \leq z \leq 5 - x + y, \}$

and the vector field  $\mathbf{f}(\mathbf{x}) := (xz, yz, xyz)^T$ .

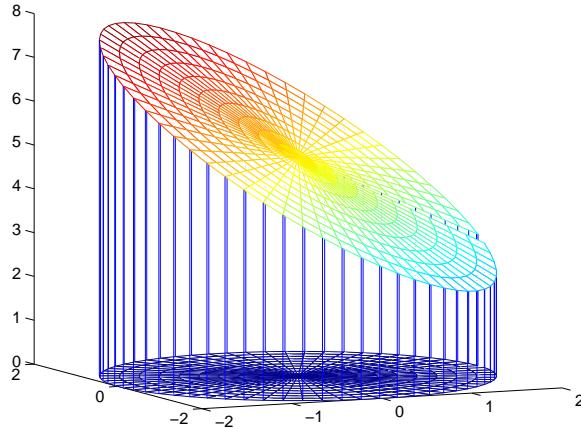
- Sketch the body  $K$  and provide the parametrizations for the three smooth surfaces  $F_1$ ,  $F_2$  and  $F_3$ , which bound  $K$ .
- Compute the volume integral  $\int_K \operatorname{div} \mathbf{f} d\mathbf{x}$ .
- Compute the flow of  $\mathbf{f}$  through the surfaces  $F_1$ ,  $F_2$  and  $F_3$

**Solution:**

a)  $x^2 + y^2 \leq 4$  : Cylinder with radius 2, axis =  $z$ -axis

Bounded from below by  $z \geq 0$  and  $x - y$ -plane

Bounded from above by plane  $z = 5 - x + y$



Parametrization  $F_1$  = bottom: a disk in  $x - y$ -plane centered at origin

$$p_1(r, \phi) = \begin{pmatrix} r \cos \phi \\ r \sin \phi \\ 0 \end{pmatrix}, \quad r \in [0, 2], \phi \in [0, 2\pi]$$

Parametrization  $F_2$  = lateral surface (sides):  $x^2 + y^2 = 4, 0 \leq z \leq 5 - x + y$

$$p_2(\phi, z) = \begin{pmatrix} 2 \cos \phi \\ 2 \sin \phi \\ z \end{pmatrix}, \quad \phi \in [0, 2\pi], z \in [0, 5 - 2 \cos \phi + 2 \sin \phi]$$

Parametrization  $F_3$  = top:

Projection on  $x - y$ -plane = disk,  $z = 5 - x + y$

$$p_3(r, \phi) = \begin{pmatrix} r \cos \phi \\ r \sin \phi \\ 5 - r \cos \phi + r \sin \phi \end{pmatrix}, \quad r \in [0, 2], \phi \in [0, 2\pi]$$

b)  $K : x = r \cos \phi, y = r \sin \phi, r \in [0, 2], \phi \in [0, 2\pi],$

$$0 \leq z \leq 5 - r \cos \phi + r \sin \phi$$

$$\operatorname{div} \mathbf{f}(x, y, z) = z + z + xy$$

$$\begin{aligned} \int_K \operatorname{div} \mathbf{f} d(x, y, z) &= \int_0^2 \int_0^{2\pi} \int_0^{5-r \cos \phi + r \sin \phi} (2z + r^2 \sin \phi \cos \phi) \cdot r dz d\phi dr \\ &= \int_0^2 \int_0^{2\pi} r(5 - r \cos \phi + r \sin \phi)^2 + r^3(5 - r \cos \phi + r \sin \phi) \sin \phi \cos \phi d\phi dr \\ &= \int_0^2 \int_0^{2\pi} r(25 - 10r \cos \phi + 10r \sin \phi + r^2 - 2r^2 \sin \phi \cos \phi) d\phi dr \\ &\quad + \int_0^2 r^3 \int_0^{2\pi} \frac{5}{2} \sin(2\phi) - r \cos^2 \phi \sin \phi + r \sin^2 \phi \cos \phi d\phi dr \\ &= 2\pi \int_0^2 25r + r^3 dr = 2\pi(50 + 4) = 108\pi \end{aligned}$$

c) The flow through  $F_1$ :

$$\frac{\partial p_1}{\partial r} \times \frac{\partial p_1}{\partial \phi} = \begin{pmatrix} \cos \phi \\ \sin \phi \\ 0 \end{pmatrix} \times \begin{pmatrix} -r \sin \phi \\ r \cos \phi \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}$$

The outer normal points downward. Hence we choose

$$\mathbf{n}_1 = \frac{\partial p_1}{\partial \phi} \times \frac{\partial p_1}{\partial r} = \begin{pmatrix} 0 \\ 0 \\ -r \end{pmatrix}$$

$$\mathbf{f}(\mathbf{x}) := (xz, yz, xyz)^T.$$

$$\mathbf{f}(p_1(r, \phi)) = \mathbf{f}(r \cos \phi, r \sin \phi, 0) = (0, 0, 0)^T$$

$$\langle f(p_1(r, \phi)), \mathbf{n}_1 \rangle = 0 \implies \int_{F_1} \mathbf{f} dO = 0$$

There is no flow through the bottom!

The flow through  $F_2$ :

$$\frac{\partial p_2}{\partial \phi} \times \frac{\partial p_2}{\partial z} = \begin{pmatrix} -2 \sin \phi \\ 2 \cos \phi \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \cos \phi \\ 2 \sin \phi \\ 0 \end{pmatrix}$$

Outer normal points outwards. So the sign of the cross product is correct!

$$\mathbf{f}(\mathbf{x}) := (xz, yz, xyz)^T.$$

$$\mathbf{f}(p_2(\phi, z) = \mathbf{f}(2\cos\phi, 2\sin\phi, z) = (2z\cos\phi, 2z\sin\phi, \dots)^T$$

$$\langle f(p_2(\phi, z)), \mathbf{n}_2 \rangle = 4z\cos^2(\phi) + 4z\sin^2(\phi) = 4z$$

$$\begin{aligned} \int_{F_2} \mathbf{f} dO &= \int_0^{2\pi} \int_0^{5-2\cos\phi+2\sin\phi} 4z dz d\phi = \int_0^{2\pi} [2z^2]_0^{5-2\cos\phi+2\sin\phi} d\phi \\ &= 2 \int_0^{2\pi} 25 - 20\cos\phi + 20\sin\phi + 4(\sin\phi - \cos\phi)^2 d\phi \\ &= 100\pi + 2 \int_0^{2\pi} 4(\sin^2\phi - 2\sin\phi\cos\phi + \cos^2\phi)\phi d\phi = 100\pi + 16\pi \end{aligned}$$

The flow through  $F_3$  can be calculated as the difference between the total flow from part b) and the flow through  $F_1$  and  $F_2$ :

$$\int_{F_3} \mathbf{f} dO = 108\pi - \int_{F_2} \mathbf{f} dO - \int_{F_1} \mathbf{f} dO = -8\pi$$

One could as well calculate the flow through  $F_1$  and  $F_3$  first, in order to then calculate the flow through  $F_2$  with the help of part b). In this case the flow through  $F_3$  can be calculated again for control purposes.

$$p_3(r, \phi) = (r\cos\phi, r\sin\phi, 5 - r\cos\phi + r\sin\phi)^T$$

$$\frac{\partial p_3}{\partial r} \times \frac{\partial p_3}{\partial \phi} = \begin{pmatrix} \cos\phi \\ \sin\phi \\ -\cos\phi + \sin\phi \end{pmatrix} \times \begin{pmatrix} -r\sin\phi \\ r\cos\phi \\ r\sin\phi + r\cos\phi \end{pmatrix} = \begin{pmatrix} r \\ -r \\ r \end{pmatrix}$$

Outer normal points upwards. So the sign of the cross product is correct!

$$\mathbf{f}(\mathbf{x}) := (xz, yz, xyz)^T.$$

$$\begin{aligned} \mathbf{f}(p_3(r, \phi)) &= (5 - r\cos\phi + r\sin\phi)(r\cos\phi, r\sin\phi, r^2\cos\phi \cdot \sin\phi)^T \\ \langle f(p_3(r, \phi)), \mathbf{n}_3 \rangle &= r(5 - r\cos\phi + r\sin\phi)(r^2\cos\phi \cdot \sin\phi + r\cos\phi - r\sin\phi) \\ \langle f(p_3(r, \phi)), \mathbf{n}_3 \rangle &= r^2 \left( \frac{5}{2}r\sin(2\phi) + 5(\cos\phi - \sin\phi) + r\sin(2\phi) - r \right. \\ &\quad \left. - r^2\cos^2\phi \cdot \sin\phi - r^2\cos\phi \cdot \sin^2\phi \right) \end{aligned}$$

$$\begin{aligned} \int_{F_3} \mathbf{f} dO &= \int_0^2 \int_0^{2\pi} -r^3(1 + r\cos^2\phi \cdot \sin\phi - r\cos\phi \cdot \sin^2\phi) d\phi dr \\ &= \int_0^2 -r^3 \left( \phi - r \frac{\cos^3\phi}{3} - \frac{\sin^3\phi}{3} \right)_{0}^{2\pi} dr = -2\pi \frac{2^4}{4} = -8\pi \end{aligned}$$