

Analysis III

for Engineering Students

Work sheet 6

Exercise 1:

a) Given are the below described body $K \subset \mathbb{R}^3$ and the vector field \mathbf{f} :

$$K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid x \in [1, 3], \quad 1 - x \leq y \leq 2 + x, \quad x^2 + y^2 - 1 \leq z \leq x^2 + y^2 + 1 \right\},$$

$$\mathbf{f} : \mathbb{R}^3 \rightarrow \mathbb{R}^3, \quad \mathbf{f}(x, y, z) = \begin{pmatrix} yz + y \\ x(z+1) + y \\ y(z+2) + x \end{pmatrix}.$$

Compute

$$\int_K \operatorname{div}(\mathbf{f}(x, y, z)) d(x, y, z).$$

b) Compute the integral

$$\int_D (1 - x^2) d(x, y)$$

over the annulus

$$D := \{(x, y)^T \in \mathbb{R}^2; 1 \leq x^2 + y^2 \leq 4\}.$$

Hint: $\cos^2 \phi = \frac{1}{2} (\cos(2\phi) + 1)$.

Solution:

a) $\operatorname{div} \mathbf{f}(x, y, z) = (f_1)_x + (f_2)_y + (f_3)_z = 0 + 1 + y.$

$$\begin{aligned} \int_K \operatorname{div} \mathbf{f}(x, y, z) d(x, y, z) &= \int_1^3 \int_{1-x}^{2+x} \int_{x^2+y^2-1}^{x^2+y^2+1} (1+y) dz dy dx = \int_1^3 \int_{1-x}^{2+x} [(1+y)z]_{x^2+y^2-1}^{x^2+y^2+1} dy dx \\ &= \int_1^3 \int_{1-x}^{2+x} 2(1+y) dy dx = \int_1^3 [(1+y)^2]_{1-x}^{2+x} dy \\ &= \int_1^3 (3+x)^2 - (2-x)^2 dx = \int_1^3 (9+6x+x^2 - 4+4x-x^2) dx \\ &= \int_1^3 (5+10x) dx = [5x+5x^2]_1^3 = 5(12-2) = 50. \end{aligned}$$

b)

$$\begin{aligned}
 \int_D (1 - x^2) d(x, y) &= \int_1^2 \int_0^{2\pi} (1 - r^2 \cos^2 \varphi) r d\varphi dr \\
 &= 2\pi \int_1^2 r dr - \int_1^2 r^3 \int_0^{2\pi} \frac{1}{2} \left(\underbrace{\cos(2\varphi)}_{\substack{\text{gives 0 since } \sin(2\varphi) \\ \text{is } \pi\text{-periodic}}} + 1 \right) d\varphi dr \\
 &= 3\pi - \pi \int_1^2 r^3 dr = 3\pi - \pi \cdot \frac{15}{4} = -\frac{3}{4}\pi
 \end{aligned}$$

Exercise 2:

Given the cone $K \subset \mathbb{R}^3$, $K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \leq x^2 + y^2 \leq 1, 0 \leq z \leq 4 - 4\sqrt{x^2 + y^2} \right\}$.
The cone has the constant density $\rho = 2$.

- a) Compute the mass of the cone.
- b) Compute the moment of inertia of the cone with respect to the z -axis using integration.
- c) Compute the moment of inertia of the cone with respect to an axis A , parallel to the z -axis, passing through the point $(\frac{3}{2}, 0, 0)^T$.

Solution to Exercise 2)

Transformation: $x = r \cos(\varphi)$, $y = r \sin(\varphi)$, $z = z$.

For the Jacobi matrix J of the coordinate transformation it holds $\det J = r$

For the parameters it holds: $r \in [0, 1]$, $\varphi \in [0, 2\pi]$, $z \in [0, 4 - 4r]$.

- a) For the mass of the homogeneous cone it holds:

$$m = \int_K \rho d(x, y, z) = \rho \int_K 1 d(x, y, z) = \rho \cdot \text{volume}(K)$$

If you use the formula for the cone volume $V = \frac{1}{3}\pi r^2 h$, you get directly:

$$m = 2 \frac{1}{3}\pi 1^2 \cdot 4 = \frac{8\pi}{3}.$$

If you don't know the formula or want to practice integration, perform the computation:

$$\begin{aligned} m &= \int_0^1 \int_0^{4-4r} \int_0^{2\pi} \rho \cdot r d\varphi dz dr = 2 \cdot \int_0^1 \int_0^{4-4r} r [\varphi]_0^{2\pi} dz dr \\ &= 4\pi \cdot \int_0^1 r [z]_0^{4-4r} dr = 4\pi \int_0^1 4r - 4r^2 dr = 4\pi \left[2r^2 - \frac{4}{3}r^3 \right]_0^1 = \frac{8\pi}{3}. \end{aligned}$$

- b) For the distance $a(x, y, z)$ of the point $(x, y, z)^T$ to z -axis one obtains:

$$a(x, y, z)^2 = x^2 + y^2 = r^2.$$

For the moment of inertia we are looking for, we compute:

$$\begin{aligned} \Theta_z &= \int_0^1 \int_0^{4-4r} \int_0^{2\pi} \rho \cdot r^2 \cdot r d\varphi dz dr = 2 \cdot \int_0^1 \int_0^{4-4r} r^3 [\varphi]_0^{2\pi} dz dr \\ &= 4\pi \cdot \int_0^1 r^3 [z]_0^{4-4r} dr = 4\pi \int_0^1 4r^3 - 4r^4 dr = 4\pi \left[r^4 - \frac{4}{5}r^5 \right]_0^1 = \frac{4\pi}{5}. \end{aligned}$$

c) Following Steiner's theorem, we have:

$\Theta_A = \Theta_z + m \cdot d^2$, where d is the distance from the axis A to z -axis. Therefore, we have

$$\Theta_A = \frac{4\pi}{5} + \frac{8\pi}{3} \cdot \frac{3^2}{2^2} = \frac{34\pi}{5}.$$

Discussion: 10.01–14.01.22