Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
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Analysis III for Engineering Students

Work sheet 5

Exercise 1:

Given the following optimization problem:

$$\begin{aligned} \text{Find the minima of} \quad & f(x,y) = 2 - x + \frac{4}{9}y \\ \text{that satisfy the constraint} \quad & g(x,y) = 25 - 9x^2 - y^2 \geq 0\,. \end{aligned} \tag{1}$$

a) Are there any local minima in the interior the admissible region, i.e. of $25 - 9x^2 - y^2 > 0$? Explain your answer.

Hint: local minima in the interior of the admissible set are also the local minima of the unconstrained problem: $\min\limits_{x,y\in\mathbb{R}} \quad f(x,y) = 2 - x + \frac{4}{9}y$.

b) Find all global minima of f that satisfy the constraint

$$g(x,y) = 25 - 9x^2 - y^2 = 0$$

using the Lagrange multiplier rule. First check the regularity condition.

Remark: This exercise can also be solved by eliminating one of the variables. However, in this exercise we would like to practice the new solution method on the simple example.

c) Find all global minima of the optimization problem (1) .
Hint: use a) and b).

Solution to exercise 1:

a) There is no extrema in the interior of the admissible region, since
$\operatorname{grad} f(x,y) = \left(1,\ \frac{4}{9}\right) \neq \mathbf{0}\,, \qquad \forall (x,y) \in \mathbb{R}^2\,.$

b) The regularity condition: $\operatorname{grad} g(x,y) = (-18x,\ -2y) \neq (0,0)^T$ is satisfied on the admissible set, since $(0,0)$ is not an admissible point.

The necessary condition for the (local) optimality is :

$$\operatorname{grad} f(x,y) + \lambda \operatorname{grad} g(x,y) = 0.$$

So we have the system of equations

$$\begin{aligned} -1 + \lambda\cdot(-18x) &= 0 \Longrightarrow \lambda \neq 0 \wedge x = -\tfrac{1}{18\lambda}\,, \\ \tfrac{4}{9} - +\lambda\cdot(-2y) &= 0 \Longrightarrow \lambda \neq 0 \wedge y = \tfrac{2}{9\lambda}\,, \\ 25 - 9x^2 - y^2 &= 0\,. \end{aligned}$$

Inserting the results from the first two lines in the last line we have:

$$25 - \frac{9}{18^2\lambda^2} - \frac{2^2}{9^2\lambda^2} = 0 \Longrightarrow 25 = \frac{9+4\cdot 4}{18^2}\cdot\frac{1}{\lambda^2}$$
$$25\lambda^2 = \frac{25}{18^2} \Longrightarrow \lambda = \pm\frac{1}{18}$$

We obtained two solutions:

$\lambda_1 = \frac{1}{18}$, $x_1 = -\frac{1}{18\lambda_1} = -1$, $y_1 = \frac{2}{9\lambda_1} = 4$

and

$\lambda_2 = -\frac{1}{18}$, $x_2 = -\frac{1}{18\lambda_2} = 1$, $y_2 = \frac{2}{9\lambda_2} = -4$.

The admissible set is compact. Hence minimum and maximum are attained on this set. The only candidates are P_1 and P_2.

$f(P_1) = 2 + 1 + \frac{16}{9}$, $\qquad f(P_2) = 2 - 1 - \frac{16}{9}$.

The only local minimum (and thus the global minimum) in the admissible set is in P_2.

Alternatively: For the Hessian matrix we have

$$\boldsymbol{H_x}(x,y) = \begin{pmatrix} -18\lambda & 0 \\ 0 & 2\lambda \end{pmatrix}.$$

It is negative definite for λ_1 and positive definite for λ_2 . Hence

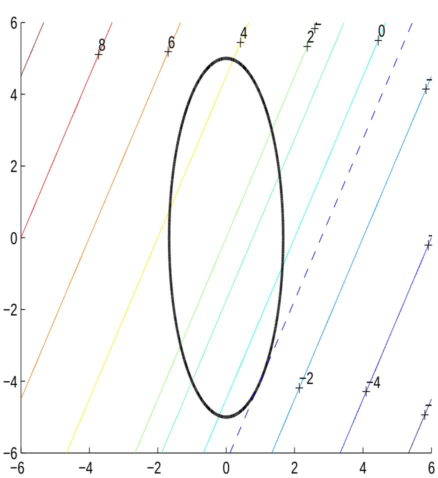
$P_1 := \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ is maximum and the point

$P_2 := \begin{pmatrix} 1 \\ -4 \end{pmatrix}$ is minimum.

c) Since

$$\left\{(x,y)^T \in \mathbb{R}^2 \,:\, g(x,y) = 25 - 9x^2 - y^2 \geq 0\,.\right\}$$

is a compact set, the function f attains on it its global minimum. But not in the interior (see a)). So the global minimum of f is on the edge. Because of b) only P_2 can be considered.



Exercise 2:

Given the minimization problem:

$$f(x,y,z) := 2x + y + z \to \min$$

subject to

$$g(x,y,z) := x^2 + y^2 + z^2 = 9.$$
$$h(x,y,z) := x^2 + (y - z)^2 = 1.$$

a) Show that $\boldsymbol{x}_0 = (1,2,2)^T$ together with the corresponding multiplier is a stationary point of the Lagrange function $F := f + \lambda_1 g + \lambda_2 h$.

b) Show that the point $\boldsymbol{x}_0 = (1,2,2)^T$ is a local maximum of the function f that fulfills the given constraint. To do this, check second order sufficient condition.

Solution sketch:

a) First order necessary optimality conditions yield

$$\begin{aligned} 2 + \lambda_1 \cdot 2x + \lambda_2 \cdot 2x &= 0 \\ 1 + \lambda_1 \cdot 2y + \lambda_2 \cdot 2(y-z) &= 0 \\ 1 + \lambda_1 \cdot 2z + \lambda_2 \cdot (-2(y-z)) &= 0 \\ x^2 + y^2 + z^2 &= 9 \\ x^2 + (y - z)^2 &= 1 \end{aligned}$$

For $x_0 = (1,2,2)^T$ we have a system

$$\begin{aligned} 2 + 2\lambda_1 + 2\lambda_2 &= 0 \\ 1 + 4\lambda_1 + \lambda_2 \cdot 0 &= 0 \\ 1 + 4\lambda_1 + \lambda_2 \cdot 0 &= 0 \\ 1 + 4 + 4 &= 9 \\ 1 + (0)^2 &= 1 \end{aligned}$$

The values $\lambda_1 = -\frac{1}{4}$ and $\lambda_2 = -\frac{3}{4}$ satisfy the system.

b) $\boldsymbol{x}_0 = (1,2,2)^T$ fulfills the given constraints. See part a).
The Hessian matrix which we have to examine is given by

$$\boldsymbol{H}_{\boldsymbol{x}}(x,y,z) = \begin{pmatrix} 2(\lambda_1+\lambda_2) & 0 & 0 \\ 0 & 2(\lambda_1+\lambda_2) & -2\lambda_2 \\ 0 & -2\lambda_2 & 2(\lambda_1+\lambda_2) \end{pmatrix}.$$

At the point $x_0 = (1,2,2)^T$ together with multipliers $\lambda_1 = -\frac{1}{4}$ and $\lambda_2 = -\frac{3}{4}$, we have

$$\boldsymbol{H}_{\boldsymbol{x}}(1,2,2) = \begin{pmatrix} -2 & 0 & 0 \\ 0 & -2 & \frac{3}{2} \\ 0 & \frac{3}{2} & -2 \end{pmatrix}$$

From the Gerschgorin theorem (from Linear Algebra ?) it follows, there is no eigenvalue of $\boldsymbol{H_x}$ greater than $-2 + \frac{3}{2}$.

Alternatively: Compute the eigenvalues.

$\mu_1 = -2$ (obtain immediately, since it is on the diagonal),

$$(-2-\mu)^2 - (\frac{3}{2})^2 = 0 \Longrightarrow \mu_{2,3} = -2 \pm \frac{3}{2}\,.$$

All eigenvalues are negative. So it is a local maximum.

Discussion: 13.12–17.12.21