Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
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Analysis III for Engineering Students

Homework sheet 5

Exercise 1

The equation

$$g(x,y) = \left(x^2 + 4y^2\right)^2 + x^2 - 4y^2 = 0$$

is an implicit description of the curve in $\mathbb{R}^2$.

a) Show that $(x,y) = (0,0)^T$ is a singular point of the implicitly defined curve

$$\left(x^2 + 4y^2\right)^2 + x^2 - 4y^2 = 0$$

and determine whether it is an isolated point, double point or a return point (cusp).

b) Show that there are no other singular points.

c) Compute the points on the curve with horizontal or vertical tangent.

Solution: [?+?+? points]

a) $g_x(x,y) = 2(x^2+4y^2)2x + 2x = 2x(2x^2+8y^2+1) \implies g_x(0,0) = 0$

$g_y(x,y) = 2(x^2+4y^2)8y - 8y = 8y(2x^2+8y^2-1) \qquad \implies g_y(0,0) = 0$

Further, it holds that $g(0,0) = 0$. So the point $(0,0)^T$ is a singular point of the curve. It holds

$$\begin{aligned} g_{xx} &= 12x^2 + 16y^2 + 2 \\ g_{xy} &= 32xy \\ g_{yy} &= 16x^2 + 192y^2 - 8 \end{aligned}$$

so we have

$$Hg(0,0) = \begin{pmatrix} +2 & 0 \\ 0 & -8 \end{pmatrix}$$

Hence, it is a double point.

b) We are looking for the points where holds $g = g_x = g_y = 0$.

$0 = g_x(x,y) = 2x(2x^2+8y^2+1) \iff x = 0$

$0 = g_y = 8y(2x^2 + 8y^2 - 1)$

$\iff y = 0 \text{ or } 2x^2 + 8y^2 = 1$

By setting $x = 0$ in $2x^2 + 8y^2 = 1$, we obtain $y^2 = 1/8$. But now it holds $g\left(0, \pm\sqrt{\frac{1}{8}}\right) = \frac{1}{4} - \frac{1}{2} \neq 0$.

So the points $\left(0, \pm\sqrt{\frac{1}{8}}\right)^T$ do not belong the curve. Hence, the point $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ is the only stationary point.

c) Points with horizontal tangent: It should hold $g = g_x = 0,\ g_y \neq 0$.

We already showed above that: $g_x = 0 \implies x = 0$.

The points we are looking for should belong to the curve:

$g(0, y) = (4y^2)^2 - 4y^2 = 0 \implies y = 0 \text{ or } 4y^2 = 1$.

$y = 0$ gives us a singular point. So we have $y = \pm\frac{1}{2}$. From the part b) we know, that there is no other singular point apart from $(0, 0)$, so the curve has horizontal tangents at:

$P_1 = \binom{0}{-\frac{1}{2}}$ $\qquad$ $P_2 = \binom{0}{\frac{1}{2}}$

We still have to find the points with a vertical tangent $g = g_y = 0,\ g_x \neq 0$.

$g_y = 8y(2x^2 + 8y^2 - 1) \stackrel{!}{=} 0$

$\iff y = 0 \lor 2x^2 + 8y^2 - 1 = 0$

$y = 0$ plugged into g

$g(x, 0) = (x^2 + 0)^2 + x^2 = 0 \iff x = 0$. So it is a singular point.

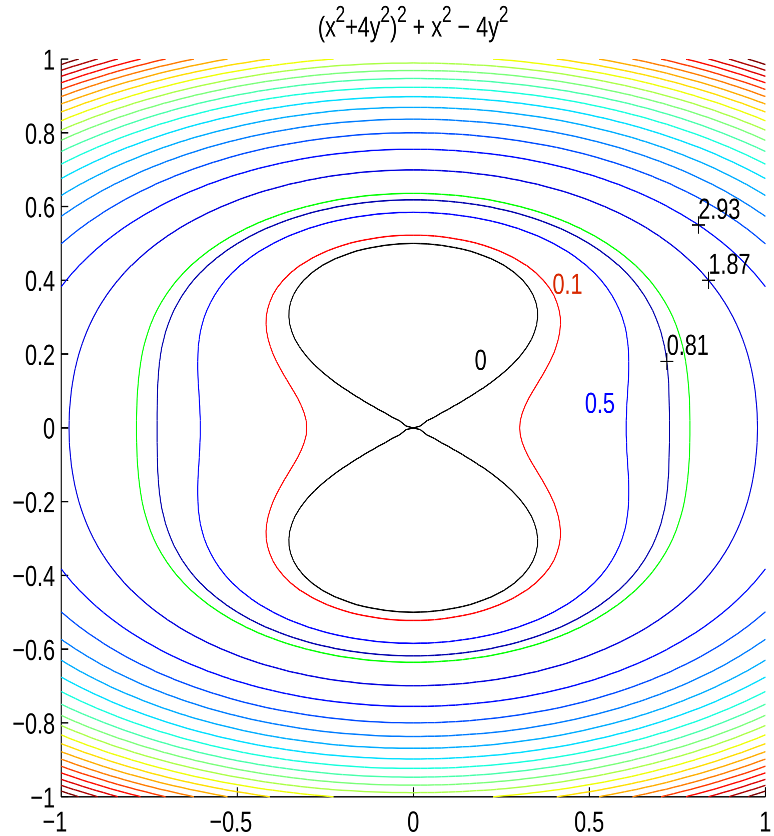
$2x^2 + 8y^2 - 1 = 0 \iff 4y^2 = \frac{1}{2} - x^2$ plugged into g

$g(x, y) = (x^2 + 4y^2)^2 + x^2 - 4y^2 = \left(x^2 + \frac{1}{2} - x^2\right)^2 + x^2 - \frac{1}{2} + x^2 = 0$.

$\iff x^2 = \frac{1}{8}, \qquad y^2 = \frac{1}{8}(1 - 2x^2) = \frac{3}{32}$

Since there is no other singular point except $(0, 0)$, the other four points in which $g = g_y = 0$ holds, are the points with vertical tangent:

$$P_3 = \begin{pmatrix} -\frac{1}{\sqrt{8}} \\ -\frac{\sqrt{3}}{2\sqrt{8}} \end{pmatrix} \qquad P_4 = \begin{pmatrix} -\frac{1}{\sqrt{8}} \\ +\frac{\sqrt{3}}{2\sqrt{8}} \end{pmatrix} \qquad P_5 = \begin{pmatrix} \frac{1}{\sqrt{8}} \\ \frac{\sqrt{3}}{2\sqrt{8}} \end{pmatrix} \qquad P_6 = \begin{pmatrix} \frac{1}{\sqrt{8}} \\ -\frac{\sqrt{3}}{2\sqrt{8}} \end{pmatrix}$$



Exercise 2: We are looking for the extrema of the function

$$f(x,y) = 2\ln\left(\frac{x}{y}\right) + x + 5y$$

that fulfill the constraint

$$g(x,y) = x\,y - 1 = 0.$$

a) Show that $(x_0, y_0)^T = (1,1)^T$ with the suitable fixed λ is a feasible stationary point of the Lagrangian $F = f + \lambda g$ and check the regularity conditions at the point $(x_0, y_0)^T = (1,1)^T$.

b) Determine of what type the stationary point $(x_0, y_0)^T = (1,1)^T$ is. To do so, assemble the Hessian matrix $\boldsymbol{H_x}\,F(x_0, y_0)$ and check its definiteness on the tangent space $\ker(Dg(x_0, y_0))$.

Solution:

a) Es gilt $g(1,1) = 1 - 1 = 0$. Hence the point $(x_0, y_0)^T = (1,1)^T$ is admissible. **[1 point]**

$\nabla g(x,y) = \begin{pmatrix} y \\ x \end{pmatrix} \Longrightarrow \nabla g(1,1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. So the regularity condition is satisfied. **[1 point]**

For f we compute

$$\nabla f(x,y) = = \begin{pmatrix} 2\,\dfrac{\frac{1}{y}}{\frac{x}{y}} + 1 \\ 2\,\dfrac{\frac{-x}{y^2}}{\frac{x}{y}} + 5 \end{pmatrix} = \begin{pmatrix} 2\,\dfrac{1}{x} + 1 \\ 2\,\dfrac{-1}{y} + 5 \end{pmatrix}. \qquad \textbf{[1 point]}$$

Thus, for an admissible stationary point of the Lagrange function $F = f + \lambda g$ we obtain the system of equations :

$$F_x = \frac{2}{x} + 1 + \lambda y = 0\,,$$

$$F_y = \frac{-2}{y} + 5 + \lambda x = 0\,,$$

$$g = xy - 1 = 0\,. \qquad \textbf{[1 point]}$$

So for $x = y = 1$

$$F_x : \frac{2}{1} + 1 + \lambda = 0 \Longleftrightarrow \lambda = -3\,,$$

$$F_y = \frac{-2}{1} + 5 + \lambda = 0 \Longleftrightarrow \lambda = -3\,,$$

$$g = 1 - 1 = 0\,. \qquad \textbf{[1 point]}$$

So $(1,1)^T$ is a stationary point of the Lagrangian with the corresponding multiplire $\lambda = -3$.

b) With $\lambda = -3$ it holds for the Hessian matrix:

$$\boldsymbol{H_x}\, F(x,y) = \begin{pmatrix} -\frac{2}{x^2} & +\lambda \\ +\lambda & \frac{2}{y^2} \end{pmatrix} \Longrightarrow \begin{pmatrix} -2 & -3 \\ -3 & 2 \end{pmatrix} \qquad \textbf{[1 point]}$$

i.e. $\boldsymbol{H_x}\, F(1,1)$ is indefinite (det $\boldsymbol{H_x}\, F(1,1) = -13$). **[1 point]**

Tangential space:

$\boldsymbol{v} = \binom{x}{y}$ with $\nabla g(1,1)^T \cdot \binom{x}{y} = 0 \Rightarrow x + y = 0$. **[1 point]**

On the tangential space:

$$(1,-1)\, \boldsymbol{H_x}\, F(1,1) \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1,-1) \begin{pmatrix} -2 & -3 \\ -3 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1,-1) \begin{pmatrix} 1 \\ -5 \end{pmatrix} = 6 > 0.$$

[1 point]

i.e. the Hessian matrix $\boldsymbol{H_x}\, F(1,1)$ is positive definite on the tangential space. Hence in the point $(1,1)$ we have a strict local minimum.
[1 point]

Exercise 3)

Compute

a) the integral

$$\int\int_{D_1} xy^2 d(x,y) \quad \text{, where } D_1 = [-1,3] \times [1,2],$$

b) the volume of the body $K \subset \mathbb{R}^3$,

$$K = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \middle| \ |x| \leq 1, \ -(1-x^2) \leq y \leq 1-x^2, \ 0 \leq z \leq \left(1 - x^2 - y\right) \right\},$$

c) and the integral

$$\int\int_{D_2} (x^2 - y^4) d(x,y) \quad \text{, where } D_2 = \{(x,y) : |x| + |y| \leq 1\}.$$

Hint: Use the symmetries!

Solution: [3+4+3 Points]

a)

$$\int\int_{D_1} xy^2 d(x,y) \quad \text{mit } D_1 = [-1,3] \times [1,2]$$

$$\begin{aligned} I_1 &:= \int_1^2 \int_{-1}^3 xy^2 \, dx \, dy = \int_1^2 y^2 \int_{-1}^3 x \, dx \, dy \\ &= \left(\int_1^2 y^2 dy \right) \left(\int_{-1}^3 x \, dx \right) = \left. \frac{y^3}{3} \right|_1^2 \cdot \left. \frac{x^2}{2} \right|_{-1}^3 = 28/3. \end{aligned}$$

b)

$$\begin{aligned} V &= \int_K 1 \, \mathrm{d}(x,y) = \int_{-1}^1 \int_{-(1-x^2)}^{1-x^2} \int_0^{1-x^2-y} \mathrm{d}z \mathrm{d}y \, \mathrm{d}x \\ &= \int_{-1}^1 \int_{-(1-x^2)}^{1-x^2} (1 - x^2 - y) \, \mathrm{d}y \, \mathrm{d}x \\ &= \int_{-1}^1 2(1-x^2)^2 - \left[\frac{1}{2} y^2 \right]_{-(1-x^2)}^{1-x^2} \mathrm{d}x \\ &= 2 \left[x - \frac{2}{3} x^3 + \frac{1}{5} x^5 \right]_{-1}^1 \\ &= 4 \left(1 - \frac{2}{3} + \frac{1}{5} \right) = \frac{32}{15}. \end{aligned}$$

c)

$$\int\int_{D_2} (x^2 - y^4) d(x,y) \quad \text{mit } D_2 = \{(x,y) : |x| + |y| \le 1\}$$

Firstly one observes that $f(x,y) = x^2 - y^4$ and D_2 are axially symmetrical with respect to both the x-axis and y-axis. So it holds $\overline{D} = D_2 \cap$ (1st quadrant) and we have

$$I_2 := \int\int_{D_2} f(x,y)\, d(x,y) \;=\; 4 \cdot \int\int_{\overline{D}} f(x,y) d(x,y)\ .$$

Hence we obtain

$$\begin{aligned}
I_2 =& 4 \int_0^1 \int_0^{1-x} x^2 - y^4 \, dy \, dx = 4 \int_0^1 \left.(x^2\, y - \frac{y^5}{5})\right|_0^{1-x} dx \\
=& 4 \int_0^1 \left(x^2(1-x) - \frac{(1-x)^5}{5} \right) dx = 4 \left[\frac{x^3}{3} - \frac{x^4}{4} + \left.\frac{(1-x)^6}{30}\right|_0^1 \right] \\
=& 4 \left[\frac{1}{3} - \frac{1}{4} + 0 - \left(0 - 0 + \frac{1}{30} \right) \right] = \frac{1}{5}.
\end{aligned}$$

Submission deadline: 13.12.–17.12.21