Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
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Analysis III for Engineering Students

Work sheet 4

Exercise 1:

a) Find an approximation to a local minimum of the function

$$f : \left[-\frac{1}{4}, \frac{1}{4}\right] \times \left[-\frac{1}{4}, \frac{1}{4}\right] \to \mathbb{R}$$

$$f(x,y) = 4x^2 + xy + 4y^2 + \sin(x-y),$$

by computing the minimum $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$ of the second-degree Taylor polynomial T_2 of f centered at the point $(0,0)^{\mathrm{T}}$.

Hint: use the sine-series.

b) Estimate the value of the remainder R_2 in the Lagrange form at the already computed point $\begin{pmatrix} \tilde{x} \\ \tilde{y} \end{pmatrix}$.

Hint: One does not need to compute every derivative exactly.

c) Show that the minimum value of f ,on the domain specified above, can not be smaller than $-\frac{9}{49}$.

Solution 1:

a) The second degree Taylor polynomial has the terms exactly up to degree 2. For the sine term one can take the Taylor polynomial of the second degree from the series

$$\sin(x-y) = (x-y) - \frac{(x-y)^3}{3!} + \frac{(x-y)^5}{5!} \mp \cdots$$

and for f has

$$T_2(x,y) = 4x^2 + xy + 4y^2 + (x-y).$$

Alternatively one can compute the second-degree Taylor polynomial T_s for $s(x,y) = \sin(x,y)$

	Value at $(0,0)^T$
$s(x,y) := \sin(x-y)$	0
$s_x(x,y) = \cos(x-y)y$	1
$s_y(x,y) = -\cos(x-y)$	-1
$s_{xx}(x,y) = -\sin(x-y)$	0
$s_{xy}(x,y) = \sin(x-y)$	0
$s_{yy}(x,y) = -\sin(x-y)$	0

and hence obtain

$T_s(x,y) = x - y$ and $T_2(x,y) = 4x^2 + xy + 4y^2 + (x-y)\,.$ **[3 points]**

grad $T_2(x,y) = (8x+y+1,\ 8y+x-1)$

grad $T_2(x,y) = 0 \iff y = -1-8x$ and $8(-1-8x)+x-1=0$

$\iff \tilde{x} = -\frac{1}{7},\ \tilde{y} = \frac{1}{7}$, is the only candidate for minimum. **[2 points]**

$H\,T_2(x,y) = \begin{pmatrix} 8 & 1 \\ 1 & 8 \end{pmatrix}$

The eigenvalues of the Hessian matrix are 7 and 9.
Hence it is a minimum. **[1 point]**

$T_2\left(-\frac{1}{7},\ \frac{1}{7}\right) = -\frac{1}{7} = -0,142857\cdots.$

b) All third derivatives have the form $\pm\, cos(x-y)$. A common upper bound for the values of the third derivatives is $C = 1$. So we have

$$\left|R_2\left(-\frac{1}{7},\ \frac{1}{7}\right)\right| \le \frac{1\cdot 2^3}{3!}\left\|\begin{pmatrix} -\frac{1}{7} \\ \frac{1}{7}\end{pmatrix} - \begin{pmatrix} 0 \\ 0\end{pmatrix}\right\|_\infty^3 = \frac{4}{21\cdot 49} < \frac{1}{5\cdot 49} \qquad \textbf{[2 Punkte]}$$

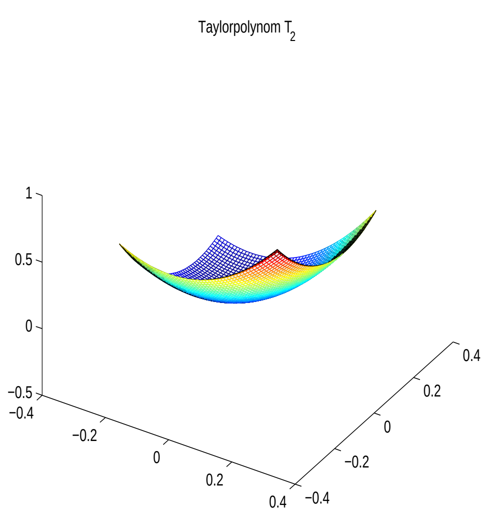
c) At the specified domain we obtain

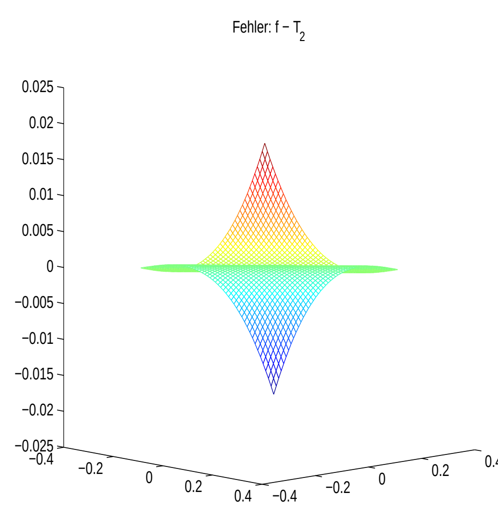
$$|R_2\,(x,\ y)| \le \frac{1\cdot 2^3}{3!}\left\|\begin{pmatrix} \frac{1}{4} \\ \frac{1}{4}\end{pmatrix} - \begin{pmatrix} 0 \\ 0\end{pmatrix}\right\|_\infty^3 = \frac{1}{48} \qquad \textbf{[1 point]}$$

For the minimum value of the function we have

$$f(x_{min}, y_{min}) \ge T_2(x_{min}, y_{min}) - \frac{1}{48} \ge T_2\left(-\frac{1}{7}, \frac{1}{7}\right) - \frac{1}{48} = -\frac{1}{7} - \frac{1}{48} \ge -\frac{7}{49} - \frac{2}{49} = -\frac{9}{49}$$

[1 point]





Exercise 2: Determine the stationary points of the following functions and check whether they are minima, maxima or saddle points:

a) $f(\boldsymbol{x}) := \boldsymbol{x}^T \boldsymbol{A}\, \boldsymbol{x} + \boldsymbol{b}^T \boldsymbol{x} + c$ with

$$\boldsymbol{x} := \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2, \quad \boldsymbol{A} := \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix}, \quad \boldsymbol{b} := \begin{pmatrix} -4 \\ 12 \end{pmatrix}, \quad c = 2018,$$

b) $g(x,y) := x^2 - xy - x + \frac{y^4}{4} + \frac{y^3}{3}$.

Solution to 2:

a)

$$f(x,y) = (x,y) \begin{pmatrix} -1 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + (-4,12) \begin{pmatrix} x \\ y \end{pmatrix} + 2018 = -x^2 + 4xy - 2y^2 - 4x + 12y + 2018.$$

$$\begin{aligned} f_x(x,y) &= -2x + 4y - 4 = 0 \iff x = 2y - 2. \\ f_y(x,y) &= 4x - 4y + 12 = 8y - 8 - 4y + 12 = 0 \\ &\iff y = -1 \implies x = -4. \end{aligned}$$

The Hessian matrix $\boldsymbol{H}(x,y) = \begin{pmatrix} -2 & 4 \\ 4 & -4 \end{pmatrix} \implies \det(\boldsymbol{H}(x,y)) = 8 - 16 < 0$ is indefinite. Hence it is a saddle point.

b) For g we have $\nabla g(x,y) = \begin{pmatrix} 2x - y - 1 \\ -x + y^3 + y^2 \end{pmatrix}$

$$2x - y - 1 = 0 \iff x = \frac{y+1}{2}$$

$$-x + y^3 + y^2 = -\frac{y+1}{2} + y^2(y+1) = \left(y^2 - \frac{1}{2}\right)(y+1) = 0$$

$$\implies y \in \left\{-1, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\}$$

So we have three stationary points:

$$\boldsymbol{P}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \qquad \boldsymbol{P}_{2,3} = \begin{pmatrix} \frac{1}{2} \mp \frac{1}{\sqrt{8}} \\ \mp \frac{1}{\sqrt{2}} \end{pmatrix}.$$

For the Hessian matrix one computes:

$$\begin{aligned} g_{xx}(x,y) &= 2 \\ g_{xy}(x,y) &= g_{yx}(x,y) = -1 \\ g_{yy}(x,y) &= 3y^2 + 2y \end{aligned}$$

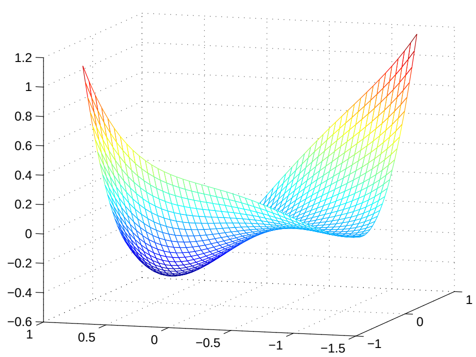
At the points $\boldsymbol{P}_1$, $\boldsymbol{P}_2$, $\boldsymbol{P}_3$ we have the following Hessian matrices

$$\boldsymbol{H}^{[1]} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \Longrightarrow \boldsymbol{H}^{[1]}_{11} = 2 > 0, \det \boldsymbol{H}^{[1]} = 2 - 1 > 0 \Longrightarrow \boldsymbol{H}^{[1]} \text{ positive definite,}$$

$$\boldsymbol{H}^{[2]} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{3}{2} - \sqrt{2} \end{pmatrix} \Longrightarrow \det \boldsymbol{H}^{[2]} = 3 - 2\sqrt{2} - 1 < 0 \Longrightarrow \boldsymbol{H}^{[2]} \text{ is indefinite,}$$

$$\boldsymbol{H}^{[3]} = \begin{pmatrix} 2 & -1 \\ -1 & \frac{3}{2} + \sqrt{2} \end{pmatrix} \Longrightarrow \boldsymbol{H}^{[3]}_{11} = 2 > 0, \det \boldsymbol{H}^{[3]} = 3 + 2\sqrt{2} - 1 > 0 \Longrightarrow \boldsymbol{H}^{[3]} \text{ positive definite.}$$

In $\boldsymbol{P}_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$ and $\boldsymbol{P}_3 = \begin{pmatrix} \frac{1}{2} + \frac{1}{\sqrt{8}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ are minima. $\boldsymbol{P}_2 = \begin{pmatrix} \frac{1}{2} - \frac{1}{\sqrt{8}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$ is a saddle point.



Discussion: 29.11 – 03.12.21