Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
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Analysis III for Engineering Students

Homework sheet 4

Exercise 1 [12 points] Given a function

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) \; = x \cdot \arctan(y) + e^{x+y} - 1\,.$$

a) Compute the second degree Taylor polynomial T_2 of f centered at a point $(0,0)^{\mathrm{T}}$.

b) Show that for the remainder $R_2(x,y) = f(x,y) - T_2(x,y)$ in the area $|x| \le 0.1\,,\ |y| \le 0.1$ the following estimate holds:

$$|R_2(x,y)| \le 0.006\,.$$

c) Find the stationary point of T_2 and check, whether it is minimum, maximum or a saddle point.

Hints: $(\arctan(y))' = \dfrac{1}{1+y^2}$, $\arctan(0) = 0\,.$

Solution:

a) **[4 points]**

	Value at $(0,0)^T$
$f(x,y) := x \cdot \arctan(y) + e^{x+y} - 1$	0
$f_x(x,y) = \arctan(y) + e^{x+y}$	1
$f_y(x,y) = \frac{x}{1+y^2} + e^{x+y}$	1
$f_{xx}(x,y) = e^{x+y}$	1
$f_{xy}(x,y) = \frac{1}{1+y^2} + e^{x+y}$	2
$f_{yy}(x,y) = \frac{-2xy}{(1+y^2)^2} + e^{x+y}$	1

$$T_2(x,y) = x + y + \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2\,.$$

b) [**4 points**]

$$\begin{array}{rclr}
 & & & \text{maximal value} \\
\hline
|f_{xxx}(x,y)| &=& |e^{x+y}| & \le\ e^{0.2} \\
|f_{xxy}(x,y)| &=& |e^{x+y}| & \le\ e^{0.2} \\
|f_{xyy}(x,y)| &=& \left|\dfrac{-2y}{(1+y^2)^2} + e^{x+y}\right| & \le\ 0.2 + e^{0.2} \\
|f_{yyy}(x,y)| &=& \left|-2x \cdot \dfrac{(1+y^2)^2 - 4y^2(1+y^2)}{(1+y^2)^4} + e^{x+y}\right| & \le\ 0.3 + e^{0.2}
\end{array}$$

For the last derivative for example one computes

$$\left|2x \cdot \frac{(1+y^2)^2 - 4y^2(1+y^2)}{(1+y^2)^4}\right| \le 0.2\left(\frac{1}{(1+y^2)^2} + \frac{4|y|^2}{(1+y^2)^3}\right) \le 0.2\left(1 + \frac{0.04}{1}\right) = 0.208\,.$$

Overall, the values of the third-order derivatives can be selected as an upper bound $C = 3 > 2.3 = \sqrt{4} + 0.3 > \sqrt{e} + 0.3 > e^{0.2} + 0.208$

Substituting into an estimate we have

$$|R_2(x,y)| \le \frac{2^3}{3!} \cdot C \cdot \|\boldsymbol{x} - \boldsymbol{x}_0\|_\infty^3 \le \frac{8}{6} \cdot 3 \cdot 0.1^3 = 0.004 < 0.006\,.$$

Note: When we estimate $e^{0.2}$ as 3 and $C = 4$, the bound is 0.00533333.

c) [**4 points**]

The stationary point $\boldsymbol{P}$ of

$$T_2(x,y) = x + y + \frac{1}{2}x^2 + 2xy + \frac{1}{2}y^2\,.$$

grad $T_2 = (1 + x + 2y,\ 1 + 2x + y) \overset{!}{=} (0,0) \implies x = y = -\frac{1}{3}$.

The Hessian matrix $\boldsymbol{H}(x,y) = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ is indefinite, because the determinant is negative.

Alternatively one can compute eigenvalues

$(1-\lambda)^2 - 4 = 0 \iff \lambda = 1 \pm 2$. Hence $\boldsymbol{P} = \begin{pmatrix} -\frac{1}{3} \\ -\frac{1}{3} \end{pmatrix}$ is a saddle point of T_2 .

Exercise 2: Given a function $f(x,y) := x^4 + y^4 + 8xy = 0$.

a) (i) Show using the implicit function theorem that $f(x,y)$ can be solved for y near the point $(x_0, y_0)^T := (2,-2)^T$. It means that there exists a function $g(x)$ with $g(2) = -2$, such that in some neighbourhood of x_0 and y_0 the following equivalence holds

$$f(x,y) = 0 \iff y = g(x).$$

(ii) Compute the first-order Taylor polynomial of function g from the part a) centered at a point $x_0 = 2$.

b) Using the implicit function theorem show that the solution set of

$$f(x,y,z) := (x^2 - 2e^{xy})z + 2 = 0$$

in a neighbourhood of the point $P_0 := (x_0, y_0, z_0)^T := (0,1,1)^T$ can be solved for x. It means that there is a function $g(y,z)$ with $g(1,1) = 0$ such that in a neighbourhood of x_0, y_0, z_0 it holds

$$f(x,y,z) = 0 \iff x = g(y,z).$$

Using the implicit function theorem for which other variable(s) one can solve the problem?

Solution sketch for Exercise 2:

a) (i) $f(2,-2) = 0$.

$$\boldsymbol{J}\,f(x,y) = \begin{pmatrix} 4x^3 + 8y \\ 4y^3 + 8x \end{pmatrix}^T \Longrightarrow \boldsymbol{J}\,f(2,-2) = \begin{pmatrix} 32 - 16 \\ -32 + 16 \end{pmatrix}^T \Longrightarrow \text{One can solve}$$

for y or for x near the point $(2,-2)^T$.

(ii) $T_1(x;2) = g(2) + g'(2)(x-2)$ For the first-order Taylor polynomial we also need $g'(2)$. Following the implicit function theorem we have

$$g'(x) = -f_x/f_y = -\frac{4x^3 + 8y}{4y^3 + 8x} \Longrightarrow g'(2) = -\frac{16}{-16} = 1.$$

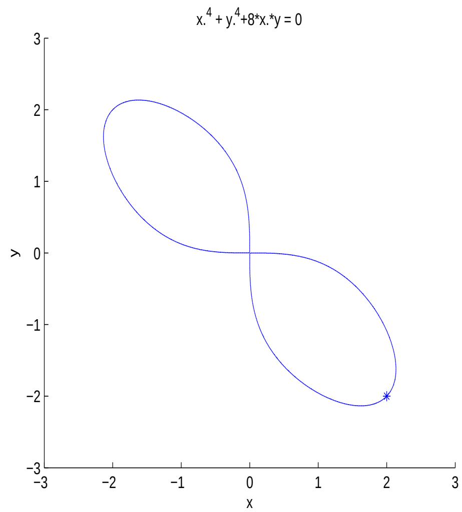
Alternatively : implicit differentiation

$$f(x,y(x)) = x^4 + (y(x))^4 + 8xy(x) = 0$$

$$f'(x,y(x)) = 4x^3 + 4y^3y' + 8y + 8xy' = (4x^3 + 8y) + (4y^3 + 8x)y' = 0$$

$$\Longrightarrow y'(x) = -\frac{4x^3 + 8y}{4y^3 + 8x}$$

$$T_1(x;2) = y(2) + y'(2)(x-2) = -2 + (x-2)$$



(iii)

b) As Jacobian matrix of f we have

$$\mathbf{J}f(x,y,z) = \big((2x - 2ye^{xy})z, \quad -2xze^{xy}, \quad x^2 - 2e^{xy}\big)$$

and hence it holds $\mathbf{J}f(0,1,1) = (-2,0,-2)$.

Since $\frac{\partial f}{\partial x} = -2$ and $\frac{\partial f}{\partial z} = -2$ as 1×1-matrices are invertible, from the implicit function theorem it follows that in some neighbourhood of P_0 there exist the functions $x(y,z)$ and $z(x,y)$ with $x(1,1) = 0$ and $f(x(y,z),y,z) = 0$ as well as $z(0,1) = 1$ and $f(x,y,z(x,y)) = 0$. The theorem does not provide the information whether it is possible to solve locally for y. An explicit form of the formula for f to y is

$y = \frac{1}{x} \cdot \ln\left(\frac{x^2}{2} + \frac{1}{z}\right)$.

This expression is not defined in any neighborhood of $x = (0,1,1)^T$!

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