

Analysis III for Engineering Students

Work sheet 3

Exercise 1:

Compute the Jacobian matrices for the following functions

- a) (i) $\hat{\mathbf{f}}_1 : \mathbb{R}^2 \rightarrow \mathbb{R}^3$, $\hat{\mathbf{f}}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + 3y \\ x^2 + y^2 \\ xy \end{pmatrix}$.
- (ii) $\mathbf{f} : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, $\mathbf{f}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} e^{(3\gamma+1)x_1} - x_2 - 1 \\ 5x_1 + e^{(3\gamma-1)x_2} - 1 \end{pmatrix}, \quad \gamma \in \mathbb{R}$ fixed parameter.
- (iii) $\tilde{\mathbf{f}} : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, $\tilde{\mathbf{f}}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -4x_1 + x_2^2 - 3x_3 \\ x_1^2 - 3x_2 + x_3^2 \\ 2x_1 - 4x_2^4 + x_3 \end{pmatrix}$.

- b) For the transformation from spherical coordinates to Cartesian coordinates

$$\mathbf{g} : \mathbb{R} \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}^3, \quad \mathbf{g}\begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} r \cos(\phi) \cos(\theta) \\ r \sin(\phi) \cos(\theta) \\ r \sin(\theta) \end{pmatrix}$$

it holds that

$$\det(\mathbf{J} \mathbf{g}(r, \phi, \theta)) = r^2 \cos(\theta).$$

Provide the determinant of the Jacobian matrix for the transformation to elliptic coordinate system

$$\tilde{\mathbf{g}} : \mathbb{R} \times [0, 2\pi] \times [-\frac{\pi}{2}, \frac{\pi}{2}] \rightarrow \mathbb{R}^3, \quad \mathbf{g}\begin{pmatrix} r \\ \phi \\ \theta \end{pmatrix} = \begin{pmatrix} ar \cos(\phi) \cos(\theta) \\ br \sin(\phi) \cos(\theta) \\ cr \sin(\theta) \end{pmatrix}$$

for fixed values of a, b, c .

Solution hint for exercise 1:

- a) (i) $J \hat{\mathbf{f}}(x, y, z) = \begin{pmatrix} 2 & 3 \\ 2x & 2y \\ y & x \end{pmatrix}$.
- (ii) $\mathbf{J} \mathbf{f}(x_1, x_2) = \begin{pmatrix} (3\gamma+1)e^{(3\gamma+1)x_1} & -1 \\ 5 & (3\gamma-1)e^{(3\gamma-1)x_2} \end{pmatrix}$.

$$(iii) \quad \mathbf{J} \tilde{\mathbf{f}}(x_1, x_2, x_3) = \begin{pmatrix} -4 & 2x_2 & -3 \\ 2x_1 & -3 & 2x_3 \\ 2 & -16x_2^3 & 1 \end{pmatrix}.$$

b) For $\mathbf{h}(x, y, z) = (ax, by, cz)$ it holds $\tilde{\mathbf{g}} = \mathbf{h} \circ \mathbf{g}$ and we have

$$\mathbf{J} \tilde{\mathbf{g}} = \mathbf{J} \mathbf{h} \cdot \mathbf{J} \mathbf{g} \text{ and } \det(\mathbf{J} \tilde{\mathbf{g}}) = \det(\mathbf{J} \mathbf{h}) \cdot \det(\mathbf{J} \mathbf{g})$$

Obviously, it holds

$$\mathbf{J} \mathbf{h} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} \quad \text{and} \quad \det(\mathbf{J} \mathbf{h}) = abc.$$

$$\det(\mathbf{J} \tilde{\mathbf{g}}) = abc \cdot r^2 \cos(\theta).$$

Exercise 2:

Compute the 2-nd order Taylor polynomial of

$$f(x, y, z) = 2 + xz + y^2 + e^x y^2 \cos(z)$$

around a point $(x_0, y_0, z_0)^T := (0, 1, \pi)^T$.

Solution:

$$f(x, y, z) = 2 + xz + y^2 + e^x y^2 \cos(z), \quad f(0, 1, \pi) = 2 + 1 + \cos(\pi) = 2.$$

$$\begin{aligned} f_x &= z + e^x y^2 \cos(z), & f_x(0, 1, \pi) &= \pi - 1 \\ f_y &= 2y + 2ye^x \cos(z), & f_y(0, 1, \pi) &= 2 - 2 = 0 \\ f_z &= x - e^x y^2 \sin(z), & f_z(0, 1, \pi) &= 0 - 0 = 0 \\ f_{xx} &= e^x y^2 \cos(z), & f_{xx}(0, 1, \pi) &= -1 \\ f_{xy} &= 2e^x y \cos(z), & f_{xy}(0, 1, \pi) &= -2 \\ f_{xz} &= 1 - e^x y^2 \sin(z), & f_{xz}(0, 1, \pi) &= 1 \\ f_{yy} &= 2 + 2e^x \cos(z), & f_{yy}(0, 1, \pi) &= 2 - 2 = 0 \\ f_{yz} &= -2ye^x \sin(z), & f_{yz}(0, 1, \pi) &= 0 \\ f_{zz} &= -e^x y^2 \cos(z), & f_{zz}(0, 1, \pi) &= 1 \end{aligned}$$

$$\begin{aligned} T(x, y, z) &= f(0, 1, \pi) + f_x(0, 1, \pi)(x - x_0) + f_y(0, 1, \pi)(y - y_0) + f_z(0, 1, \pi)(z - z_0) \\ &\quad + \frac{1}{2} (x - x_0, y - y_0, z - z_0) H f(0, 1, \pi) \begin{pmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{pmatrix} \\ &= 2 + x(\pi - 1) + \frac{1}{2} (x, y - 1, z - \pi) \begin{pmatrix} -1 & -2 & 1 \\ -2 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y - 1 \\ z - \pi \end{pmatrix} \\ &= 2 + \pi x - x + \frac{1}{2} (x, y - 1, z - \pi) \begin{pmatrix} -x - 2(y - 1) + (z - \pi) \\ -2x \\ x + (z - \pi) \end{pmatrix} \\ &= 2 + \pi x - x + \frac{1}{2} [-x^2 - 2x(y - 1) + x(z - \pi) - 2x(y - 1) + x(z - \pi) + (z - \pi)^2] \\ &= 2 + \pi x - x - \frac{x^2}{2} - 2x(y - 1) + x(z - \pi) + \frac{(z - \pi)^2}{2} \end{aligned}$$

Alternatively, one computes

$$\begin{aligned}
 T(x, y, z) &= f(0, 1, \pi) + \text{grad } f(0, 1, \pi)^T \begin{pmatrix} x \\ y - 1 \\ z - \pi \end{pmatrix} \\
 &\quad + \frac{1}{2!} (f_{xx}(0, 1, \pi)(x - 0)^2 + 2f_{xy}(0, 1, \pi)(x - 0)(y - 1) \\
 &\quad + 2f_{xz}(0, 1, \pi)(x - 0)(z - \pi) + f_{yy}(0, 1, \pi)(y - 1)^2 \\
 &\quad + 2f_{yz}(0, 1, \pi)(y - 1)(z - \pi) + f_{zz}(0, 1, \pi)(z - \pi)^2) \\
 &\quad 2 + \pi x - x - \frac{x^2}{2} - 2x(y - 1) + x(z - \pi) + \frac{(z - \pi)^2}{2}.
 \end{aligned}$$

Or a Taylor series with

$$\begin{aligned}
 \cos(z) &= -1 + \frac{1}{2}(z - \pi)^2 + \dots, & e^x &= 1 + x + \frac{x^2}{2} + \dots \\
 y^2 &= (y - 1)^2 + 2(y - 1) + 1.
 \end{aligned}$$

Classes: 15.-19.11.21