Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
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Analysis III for Engineering Students

Homework sheet 3

Exercise 1:

Given a function $f : \mathbb{R}^3 \to \mathbb{R}$, $f(\boldsymbol{x}) := 2x^2 + y^2 - 4x + z$, a point $\boldsymbol{x}_0 = (1, 2, 3)^T$, and a direction $\boldsymbol{a} = \frac{1}{\sqrt{6}}(-1, -1, -2)$:

a) Provide the equation of the level surface $N_{\boldsymbol{x}_0}$ of the function f at the point $\boldsymbol{x}_0 = (1, 2, 3)^{\mathrm{T}}$ and compute the gradient of f at $\boldsymbol{x}_0$.

b) Compute the directional derivative $D_{\boldsymbol{a}} f(\boldsymbol{x}_0)$ in the direction $\boldsymbol{a} = \frac{1}{\sqrt{6}}(-1, -1, -2)^{\mathrm{T}}$.

Can you determine whether it is a direction of ascent or descent? Can you tell whether the function values increase or decrease when one moves from $\boldsymbol{x}_0$ in the direction $\boldsymbol{a}$?

c) Compute the function values $f(\boldsymbol{x}_0 + t\,\boldsymbol{a})$ for $t = \frac{\sqrt{6}}{2}, 2\sqrt{6}, 3\sqrt{6}$.

Is there a contradiction to your result from b?

Solution: (3+ 2 + 5 Points)

a) For the level surface we are computing it holds

$$2x^2 + y^2 - 4x + z = 5,$$

$$\operatorname{grad} f(x, y) = (4x - 4, 2y, 1)^T,$$
$$\operatorname{grad} f(1, 2, 3) = (0, 4, 1)^T.$$

b) $D_{\boldsymbol{a}} f(\boldsymbol{x}_0) = \boldsymbol{a} \cdot \operatorname{grad} f(\boldsymbol{x}_0) = < \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} -1 \\ -1 \\ -2 \end{pmatrix} > = \frac{1}{\sqrt{6}}(0 - 4 - 2) = -\sqrt{6}$

From the definition of the directional derivative,

$\mathrm{D}_{\boldsymbol{a}} f(\boldsymbol{x}_0) := \frac{\partial}{\partial \boldsymbol{a}} f(\boldsymbol{x}_0) := \lim\limits_{h \to 0} \dfrac{f(\boldsymbol{x}_0 + h\,\boldsymbol{a}) - f(\boldsymbol{x}_0)}{h}$,

hence $\boldsymbol{a}$ must be the direction of the descent!

c)

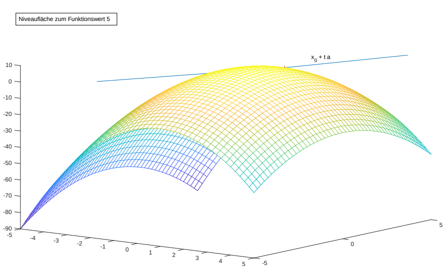
$$f(\boldsymbol{x}_0 + \frac{\sqrt{6}}{2}\boldsymbol{a}) = f\left(\begin{pmatrix}1\\2\\3\end{pmatrix} + \frac{1}{2}\begin{pmatrix}-1\\-1\\-2\end{pmatrix}\right) = f\left(\begin{pmatrix}\frac{1}{2}\\\frac{3}{2}\\2\end{pmatrix}\right) = \frac{11}{4} < 5\,.$$

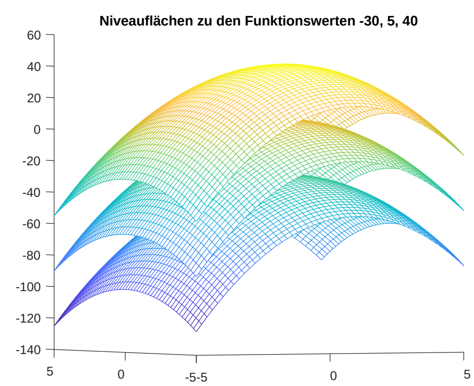
$$f(\boldsymbol{x}_0 + 2\sqrt{6}\boldsymbol{a}) = f\left(\begin{pmatrix}1\\2\\3\end{pmatrix} + 2\begin{pmatrix}-1\\-1\\-2\end{pmatrix}\right) = f\left(\begin{pmatrix}-1\\0\\-1\end{pmatrix}\right) = 5\,.$$

$$f(\boldsymbol{x}_0 + 3\sqrt{6}\boldsymbol{a}) = f\left(\begin{pmatrix}1\\2\\3\end{pmatrix} + 3\begin{pmatrix}-1\\-1\\-2\end{pmatrix}\right) = f\begin{pmatrix}-2\\-1\\-3\end{pmatrix} = 14 > 5\,.$$

Obviously, starting from $\boldsymbol{x}_0$ and moving with step of size $\frac{\sqrt{6}}{2}$ in the direction $\boldsymbol{a}$, the function value decreases. However, moving further in the direction $\boldsymbol{a}$, for the step size $2\sqrt{6}$, one again obtains the function value 5. If one moves even further away from $\boldsymbol{x}_0$, the function value rises above $f(\boldsymbol{x}_0)$.

Note: statements about ascent, descent, etc. are usually only local statements. They only apply to sufficiently small steps!





Exercise 2:

Let $\boldsymbol{u} = (u(x,y), v(x,y))^{\mathrm{T}}$ be a velocity field of the two-dimensional flow, $r = \sqrt{x^2+y^2}$ and $\epsilon \in \mathbb{R}^+$. Given the velocity fields

a) $u = \epsilon x, \quad v = \epsilon y$

b) $u = \epsilon \dfrac{x}{r^2}, \quad v = \epsilon \dfrac{y}{r^2}, \, (x,y) \neq (0,0)$ (isolated source)

c) $u = \epsilon \dfrac{-y}{r^2}, \quad v = \epsilon \dfrac{x}{r^2}, \, (x,y) \neq (0,0)$ (isolated vortex)

compute the source density $\operatorname{div} \boldsymbol{u}$ and vortex density $\operatorname{rot} \boldsymbol{u} := v_x - u_y$. Sketch the vector fields and a few associated streamlines (they are the solutions of the system of differential equations $\dot{x} = u, \, \dot{y} = v$ or the differential equation $y'(x) = v(x,y)/u(x,y)$).

Solution 2: [2+ 4+ 4 Points]

a) It holds $\frac{dy}{dx} = \frac{y}{x}$. This is a separable differential equation for y with the solution $y(x) = k \cdot x$ .The streamlines are rays emanating from the origin. They move with the speed $\| \boldsymbol{u} \| = \sqrt{(\epsilon x)^2 + (\epsilon y)^2} = \epsilon r$.

   We have immediately:

   $u_x = \epsilon, \qquad v_y = \epsilon$ and thus div $(u,v) = 2\epsilon$.

   $u_y = 0, \qquad v_x = 0$ and thus rot $(u,v) = 0$.

b) It holds again $\frac{dy}{dx} = \frac{y}{x}$ and $y(x) = k \cdot x$. The streamlines are again rays emanating from the origin. However, they are moving with the speed ϵ/r.

   One easily obtains

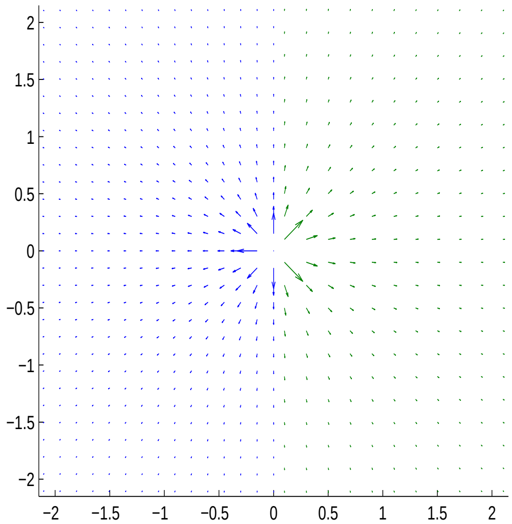
   $u_x = \epsilon \dfrac{y^2 - x^2}{r^4}, \qquad v_y = \epsilon \dfrac{x^2 - y^2}{r^4}$ and div $(u,v) = 0$.

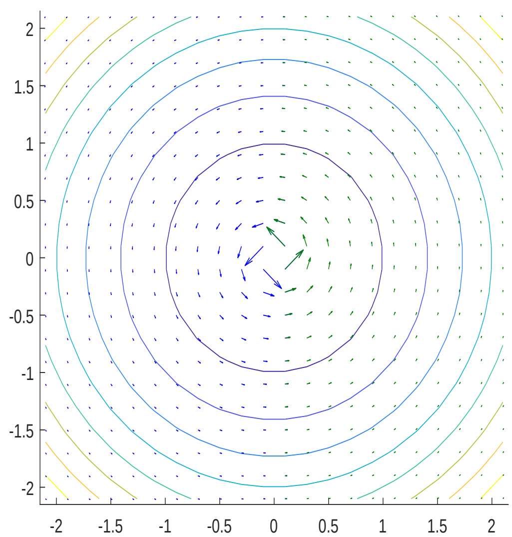
   $u_y = \epsilon \dfrac{-2xy}{r^4}, \qquad v_x = \epsilon \dfrac{-2xy}{r^4}$ and rot $(u,v) = 0$.

c) It holds $\frac{dy}{dx} = -\frac{x}{y}$. This is a separable differential equation for y with the solution $(y(x))^2 = k - x^2$. The streamlines are circles around zero. They move in a mathematically positive direction. The speed is again ϵ/r. With the exception of a sign, only the roles of u and v are exchanged, so we have

   $u_x = \epsilon \dfrac{2xy}{r^4}, \qquad v_y = \epsilon \dfrac{-2xy}{r^4}$ and hence div $(u,v) = 0$.

   $u_y = \epsilon \dfrac{-x^2 + y^2}{r^4}, \qquad v_x = \epsilon \dfrac{y^2 - x^2}{r^4}$ and thus rot $(u,v) = 0$.





Submission deadline: 15.–19.11.21