Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
Prof. Dr. I. Gasser
Dr. H. P. Kiani, S. Onyshkevych

Analysis III for Engineering Students

Work sheet 2

Exercise 1: Let $f, g : \mathbb{R}^2 \longrightarrow \mathbb{R}$.

$$f(x,y) := 3x - 5y, \qquad g(x,y) := \frac{1}{5}(x^2+y^2) + 1 .$$

a) Calculate the gradients of f and g.

b) For f draw the contour lines (level curves)
$f^{-1}(C) := \{(x,y)^{\mathrm{T}} : f(x,y) = C\}$
for the function values $C_1 = 5, C_2 = 0$ and $C_3 = -10$.
At points $P_1 = \binom{0}{-1}$, $P_2 = \binom{5}{3}$ and $P_3 = \binom{-5}{-1}$ also provide the direction of the gradient.

c) For g draw the contour lines
$g^{-1}(C) := \{(x,y)^{\mathrm{T}} : g(x,y) = C\}$
for function values $C_4 = \frac{6}{5}, C_5 = \frac{21}{5}$ and $C_6 = 6$.
At points $P_4 = \binom{0}{-1}$, $P_5 = \binom{4}{0}$ and $P_6 = \binom{3}{4}$ also provide the direction of the gradient.

d) Based on your observations (i.e. without proof), try to formulate a guess on how the direction of gradient at a given point is related to the direction of the contour line through that point.

Solution:

a) grad $f(x,y) = (3, -5)$.
grad $g(x,y) = \frac{1}{5}(2x,\, 2y)$

b) The contour lines to f are straight lines $y = \frac{3}{5}x - \frac{C}{5}$.

The contour lines to the given values are straight lines with slope $\frac{3}{5}$ through the respective points.

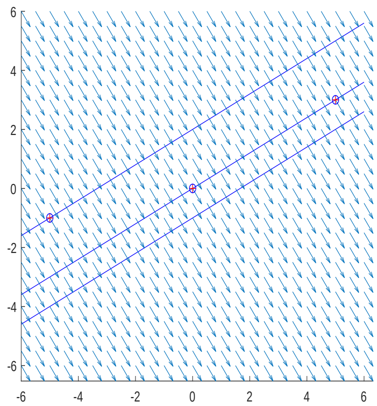
All gradients have the same direction and are orthogonal to the contour lines.

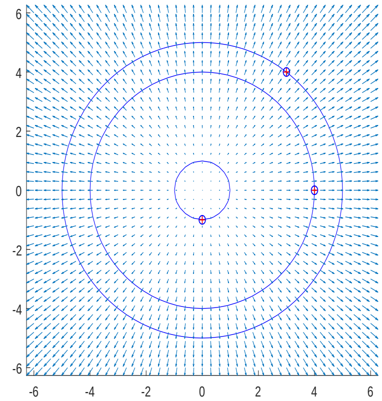
c) The contour lines to g are circles around the origin.

The contour lines to the given values are again those circles around the origin that go through the given points. That is, the circles with radii 1, 4, 5.

The gradients have the same direction as the position vectors.

d) The gradients are orthogonal to the contour lines.





Exercise 2:

Let

$$f : \mathbb{R}^2 \to \mathbb{R}, \qquad f(x,y) \;=\; \cos(2x-3y) + x^3 - y^3 + 2y^2 .$$

a) Find all first, second and third order partial derivatives of f .

b) the *tangential plane* to the graph of a differentiable function $f : D_f \longrightarrow \mathbb{R}$ at point $(x^0, y^0) \in D_f \subset \mathbb{R}^2$ is defined by

$$z \;=\; f(x^0,y^0) + f_x(x^0,y^0)\,(x-x^0) + f_y(x^0,y^0)\,(y-y^0) .$$

Give the equation of the tangential plane to the graph of f at the point $(x^0, y^0) = (\frac{\pi}{4}, 0)$.

Solution 2:

a)

$$\begin{aligned}
f(x,y) &= \cos(2x-3y) + x^3 - y^3 + 2y^2 ,\\
f_x(x,y) &= -2\sin(2x-3y) + 3x^2 ,\\
f_y(x,y) &= 3\sin(2x-3y) - 3y^2 + 4y ,\\
f_{xx}(x,y) &= -4\cos(2x-3y) + 6x ,\\
f_{xy}(x,y) &= f_{yx}(x,y) = 6\cos(2x-3y) ,\\
f_{yy}(x,y) &= -9\cos(2x-3y) - 6y + 4 ,\\
f_{xxx}(x,y) &= 8\sin(2x-3y) + 6 ,\\
f_{xxy}(x,y) &= f_{xyx}(x,y) = f_{yxx}(x,y) = -12\sin(2x-3y) ,\\
f_{xyy}(x,y) &= f_{yxy}(x,y) = f_{yyx}(x,y) = 18\sin(2x-3y) ,\\
f_{yyy}(x,y) &= -27\sin(2x-3y) - 6 .
\end{aligned}$$

b) $f(x,y) = \cos(2x - 3y) + x^3 - y^3 + 2y^2, \qquad f(\frac{\pi}{4}, 0) = \frac{\pi^3}{64}\,,$

$f_x(x,y) = -2\sin(2x - 3y) + 3x^2, \qquad f_x(\frac{\pi}{4}, 0) = -2 + \frac{3\pi^2}{16}\,,$

$f_y(x,y) = 3\sin(2x - 3y) - 3y^2 + 4y, \qquad f_y(\frac{\pi}{4}, 0) = 3\,,$

Tangential plane: $z = f(x_0, y_0) + f_x(x_0, y_0) \cdot (x - x_0) + f_y(x_0, y_0) \cdot (y - y_0)\,.$

$z = \frac{\pi^3}{64} + \frac{3\pi^2 - 32}{16}\,(x - \frac{\pi}{4}) + 3y\,.$

Classes: 01.–05.11.21