

# Analysis III

## for Engineering Students

### Homework sheet 2

**Problem 1:**

a) Find all first and second order partial derivatives of

$$s(x, y, z) := xyz \sin(x + y + z) \quad \text{and} \quad g(x, y, z) := \frac{\cos^2(x)e^y}{z}.$$

b) Calculate for the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$

$$f(x, y, z) = \arctan(x)e^y + \sin(x) \ln(1 + y^2)z + x^2 e^{z^2}$$

the derivative  $f_{xyz}$  as well as  $\nabla f(x, y, z)$ .

**Solution 1:**

a)

$$s(x, y, z) := xyz \sin(x + y + z),$$

$$s_x(x, y, z) = yz \sin(x + y + z) + xyz \cos(x + y + z)$$

$$s_{xx}(x, y, z) = 2yz \cos(x + y + z) - xyz \sin(x + y + z)$$

$$s_{xy}(x, y, z) = (z - xyz) \sin(x + y + z) + (xz + yz) \cos(x + y + z)$$

All other derivatives we get immediately due to the symmetry, since variables  $x, y, z$  are interchangeable. For example, one gets by exchanging the roles of  $x$  and  $z$ :

$$s_{zy}(x, y, z) = s_{yz}(x, y, z) = (x - xyz) \sin(x + y + z) + (xz + yx) \cos(x + y + z)$$

Or for the calculation of  $s_{yy}$  by exchanging  $x$  and  $y$  in  $s_{xx}$

$$s_{yy}(x, y, z) = 2xz \cos(x + y + z) - xyz \sin(x + y + z)$$

For the calculation of  $f_{xz}$  one exchanges  $y$  and  $z$  in  $f_{xy}$ :

$$s_{xz}(x, y, z) = (y - xyz) \sin(x + y + z) + (xy + yz) \cos(x + y + z)$$

and so on.

A differentiation of  $g(x, y, z) = \frac{\cos^2(x)e^y}{z}$  with respect to  $y$  does not change the function. Therefore:

$$g_x(x, y, z) = \frac{-2 \cos(x) \sin(x) e^y}{z}$$

$$g_y(x, y, z) = g(x, y, z) = \frac{\cos^2(x) e^y}{z}$$

$$g_z(x, y, z) = g_{zy}(x, y, z) = -\frac{\cos^2(x) e^y}{z^2}.$$

$$g_{xx}(x, y, z) = \frac{(-2 \cos^2(x) + 2 \sin^2(x)) e^y}{z}, \quad g_{xy}(x, y, z) = g_x(x, y, z),$$

$$g_{xz}(x, y, z) = \frac{2 \cos(x) \sin(x) e^y}{z^2}, \quad g_{yx}(x, y, z) = g_x(x, y, z),$$

$$g_{yy}(x, y, z) = g(x, y, z), \quad g_{yz}(x, y, z) = g_z(x, y, z),$$

$$g_{zx}(x, y, z) = g_{xz}(x, y, z), \quad g_{zy}(x, y, z) = g_z(x, y, z),$$

$$g_{zz}(x, y, z) = \frac{2 \cos^2(x) e^y}{z^3}$$

b) In order to calculate the third order derivative  $f_{xyz}$  of

$$f(x, y, z) = \arctan(x) e^y + \sin(x) \ln(1 + y^2) z + x^2 e^{z^2}$$

it makes sense to differentiate with respect to  $y$  or  $z$  first. For example:

$$f_z(x, y, z) = 0 + \sin(x) \ln(1 + y^2) + 2zx^2 e^{z^2}$$

$$f_{yz}(x, y, z) = \sin(x) \frac{2y}{1 + y^2} + 0$$

$$f_{xyz}(x, y, z) = \frac{2y \cos(x)}{1 + y^2}.$$

$$\nabla f(x, y) = \begin{pmatrix} f_x(x, y, z) \\ f_y(x, y, z) \\ f_z(x, y, z) \end{pmatrix} = \begin{pmatrix} \frac{e^y}{1+x^2} + \cos(x) \ln(1 + y^2) z + 2xe^{z^2} \\ \arctan(x) e^y + \sin(x) \frac{2yz}{1+y^2} \\ \sin(x) \ln(1 + y^2) + 2zx^2 e^{z^2} \end{pmatrix}.$$

**Problem 2:** The function

$$u(x, t) := \frac{1}{2} \left[ \sin\left(\frac{2\pi}{L}(x + ct)\right) + \sin\left(\frac{2\pi}{L}(x - ct)\right) \right]$$

describes approximately the displacement of the point  $x \in [0, L]$  of a vibrating string of length  $L$  at time  $t > 0$

The position and the velocity of the string at time  $t = 0$  are  $u(x, 0) = \sin\left(\frac{2\pi x}{L}\right)$  and  $u_t(x, 0) = 0$ . These are the so-called initial values.

a) Calculate the displacement at the end points of the string, the so-called boundary values  $u(0, t)$  and  $u(L, t)$ .

b) Show that  $u$  satisfies the wave equation  $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ .

c) Try to sketch the form of the string for  $t = 0, \frac{L}{6c}, \frac{L}{4c}, \frac{L}{3c}, \frac{L}{2c}, \frac{L}{c}$ .

Hint:  $\sin(a + b) + \sin(a - b) = 2 \sin(a) \cos(b)$ .

**Solution 2:**

a)  $u(0, t) = u(L, t) = 0$ .

b) Calculate derivatives and substitute into the equation.

$$\begin{aligned} u_x(x, t) &= \frac{1}{2} \cdot \frac{2\pi}{L} \left[ \cos\left(\frac{2\pi}{L}(x + ct)\right) + \cos\left(\frac{2\pi}{L}(x - ct)\right) \right] \\ u_{xx}(x, t) &= \frac{\pi}{L} \cdot \frac{2\pi}{L} \left[ -\sin\left(\frac{2\pi}{L}(x + ct)\right) - \sin\left(\frac{2\pi}{L}(x - ct)\right) \right] \\ u_t(x, t) &= \frac{1}{2} \cdot \frac{2c\pi}{L} \left[ \cos\left(\frac{2\pi}{L}(x + ct)\right) - \cos\left(\frac{2\pi}{L}(x - ct)\right) \right] \\ u_{tt}(x, t) &= \frac{c\pi}{L} \cdot \frac{2c\pi}{L} \left[ -\sin\left(\frac{2\pi}{L}(x + ct)\right) - \sin\left(\frac{2\pi}{L}(x - ct)\right) \right] = c^2 u_{xx}(x, t) \end{aligned}$$

c) Using the hint  $\sin(a + b) + \sin(a - b) = 2 \sin(a) \cos(b)$  we get

$$u(x, t) := \frac{1}{2} \left[ \sin\left(\frac{2\pi}{L}(x + ct)\right) + \sin\left(\frac{2\pi}{L}(x - ct)\right) \right] = \sin\left(\frac{2x\pi}{L}\right) \cdot \cos\left(\frac{2c\pi t}{L}\right)$$

$$u(x, 0) = \sin\left(\frac{2\pi}{L}x\right) \cos(0) = \sin\left(\frac{2\pi}{L}x\right).$$

$$u(x, \frac{L}{6c}) = \sin\left(\frac{2x\pi}{L}\right) \cdot \cos\left(\frac{2c\pi L}{6cL}\right) = \sin\left(\frac{2\pi x}{L}\right) \cdot \cos\left(\frac{\pi}{3}\right) = \frac{1}{2} u(x, 0)$$

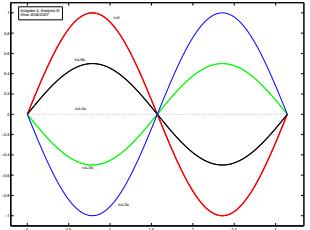
Similarly one obtains:

$$u(x, \frac{L}{4c}) = \sin\left(\frac{2\pi x}{L}\right) \cdot \cos\left(\frac{2\pi}{L} \cdot c \cdot \frac{L}{4c}\right) = \sin\left(\frac{2\pi x}{L}\right) \cdot \cos\left(\frac{\pi}{2}\right) = 0$$

$$u(x, \frac{L}{3c}) = \sin\left(\frac{2\pi x}{L}\right) \cdot \cos\left(\frac{2\pi}{3}\right) = -\frac{1}{2}u(x, 0)$$

$$u(x, \frac{L}{2c}) = \sin\left(\frac{2\pi x}{L}\right) \cdot \cos(\pi) = -u(x, 0)$$

$$u(x, \frac{L}{c}) = \sin\left(\frac{2\pi x}{L}\right) \cdot \cos(2\pi) = u(x, 0).$$



**Due date:** 01.-05.11.21