Fachbereich Mathematik der Universität Hamburg **WiSe 2021/22**
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Analysis III for Engineering Students

Work sheet 1

Exercise 1: Consider the following sets

$$M_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x, y \in \mathbb{R},\; x^2 + y^2 \leq 1 \right\},$$

$$M_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x, y \in \mathbb{R},\; x^2 + y^2 < 4 \right\},$$

$$M_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x, y \in \mathbb{R},\; 1 \leq x^2 + y^2 < 4 \right\},$$

$$M_4 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;:\; x, y, z \in \mathbb{R},\; x^2 + y^2 \leq 1 \right\},$$

$$M_5 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;:\; x, y, z \in \mathbb{R},\; x^2 + y^2 + z^2 < 1 \right\},$$

$$M_6 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;:\; (x, y) \cdot (1, 2)^T = 1 \right\},$$

$$M_7 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \;:\; (x, y, z) \cdot (1, 2, 1)^T < 1 \right\},$$

$$M_8 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;:\; x, y, z \in \mathbb{R},\; z = x^2 + y^2 \right\}.$$

$$M_9 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x, y \in \mathbb{R},\; (x+3)^2 + y^2 \leq 1 \right\} \cup \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \;:\; x, y \in \mathbb{R},\; (x-3)^2 + y^2 \leq 1 \right\}.$$

a) Which are the boundary points of $M_1, \ldots, M_9$?

b) Decide for each set $M_1, \ldots, M_9$ if it is closed, open or neither closed nor open.

c) Which of the sets $M_1, \ldots, M_9$ are bounded?

d) Which sets $M_1, \ldots, M_9$ are connected? Which are convex?

Solution 1:

a)

$$\partial M_1 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ x^2 + y^2 = 1 \right\} \qquad \text{circle } C_1,\ \text{radius 1, center 0,}$$

$$\partial M_2 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ x^2 + y^2 = 4 \right\} \qquad \text{circle } C_2,\ \text{radius 2, center 0,}$$

$$\partial M_3 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ x^2 + y^2 \in \{1, 4\} \right\} \quad \text{two circles } C_1 \text{ and } C_2,$$

$$\partial M_4 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R},\ x^2 + y^2 = 1 \right\} \qquad \text{right circular cylinder side,}$$

$$\partial M_5 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R},\ x^2 + y^2 + z^2 = 1 \right\} \text{sphere, surface of a ball, radius 1, center 0,}$$

$$\partial M_6 := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \ : \ (x, y) \cdot (1, 2)^T = 1 \right\} \qquad \text{line } x + 2y = 1,$$

$$\partial M_7 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ (x, y, z) \cdot (1, 2, 1)^T = 1 \right\} \quad \text{plane: } x + 2y + z = 1,$$

$$\partial M_8 := \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \ : \ x, y, z \in \mathbb{R},\ z = x^2 + y^2 \right\} \qquad \text{paraboloid,}$$

$$\begin{aligned} \partial M_9 := & \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ (x + 3)^2 + y^2 = 1 \right\} \\ \cup & \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \ : \ x, y \in \mathbb{R},\ (x - 3)^2 + y^2 = 1 \right\} \text{ two circles with radius 1 and centers } (\mp 3, 0)^T. \end{aligned}$$

b) $M_1 :$ closed disc with radius 1 and center zero.

$M_2 :$ open disc with radius 2 and center zero.

$M_3 :$ Annulus, region between two concentric circles with centres $= 0$. The circle with radius 1 belongs to M_3 whereas the circle with radius 2 does not belong to M_3. Neither open nor closed.

$M_4 :$ Closed infinitely long cylinder. Note: the complement is open!

$M_5 :$ Open Ball, radius 1, centre zero.

M_6 : Line in $\mathbb{R}^2$, closed.

M_7 : Half-space in $\mathbb{R}^3$ without the dividing plane, hence open.

M_8 : Plane in $\mathbb{R}^3$, closed. The complement is open!

M_9 : Two closed discs with radius 1 and centres $(\mp 3, 0)^T$. Closed.

c) The sets M_1, M_2, M_3, M_5 and M_9 are bounded. If we choose $r \in \mathbb{R}$ large enough the sets are contained in a ball B_r with radius r and centre zero.

The sets M_4, M_6, M_7 and M_8 are unbounded. There is no $r \in \mathbb{R}$ with

$M_k \subset B_r,\ k \in \{4, 6, 7, 8\}$.

d) All sets accept M_9 are connected: Any two points belonging to one of the sets $M_k, k \neq 9$ can be connected via a curve lying in M_k . This is not true for M_9 . Consider for example $(-2, 0)^T$ and $(2, 0)^T$.

Since any convex set is also connected, M_9 is not convex.

M_3 is not convex. Consider for example the line segment connecting $(-1, 0)^T$ and $(1, 0)^T$.

M_8 is not convex. Example: the line segment connecting $(-1, 0, 1)^T$ and $(1, 0, 1)^T$ does not completely belong to M_8 .

All the other sets are convex.

Exercise 2: Consider the functions $f_k : \mathbb{R}^2 \to \mathbb{R},\ k = 1, 2, 3, 4$

a) $f_1(x, y) = 2x + 3y,$ b) $f_2(x, y) = x^2 + \frac{y^2}{9},$

c) $f_3(x, y) = \cos(x - y^2)\,,$ d) $f_4(x, y) = \exp(x \cdot y)\,.$

Draw a few contour lines (curves along which the function has a constant value)

$$f_k^{-1}(C) := \{(x, y)^{\mathrm{T}} : f(x, y) = C\}$$

for each f_k.

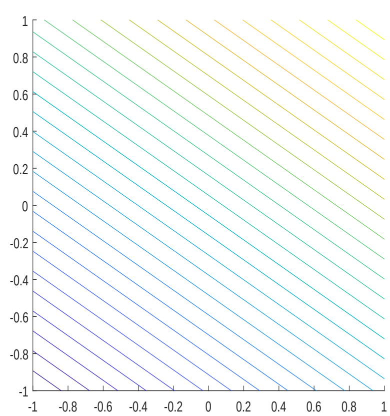
Solution 2:

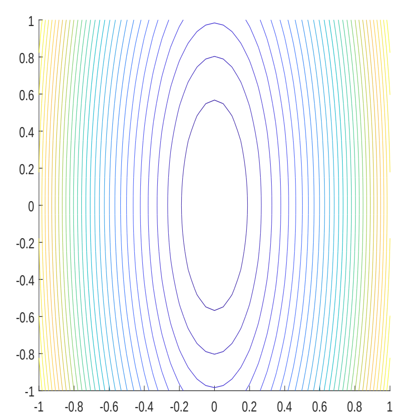
a) $f_1 : 2x + 3y = C \longrightarrow$

the contour lines are parallel lines $y = \frac{C-2x}{3}$.

b) $f_2 : x^2 + \frac{y^2}{9}$

the contour lines are ellipses with centre zero. The axis in y-direction is three times as long as the axis in x-direction.





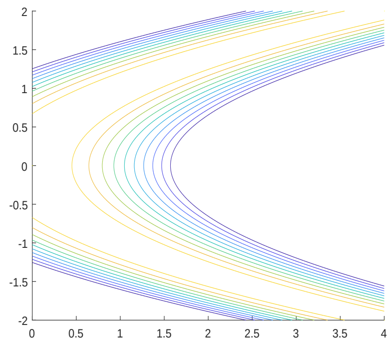
c) $f_3 : \cos(x - y^2)$

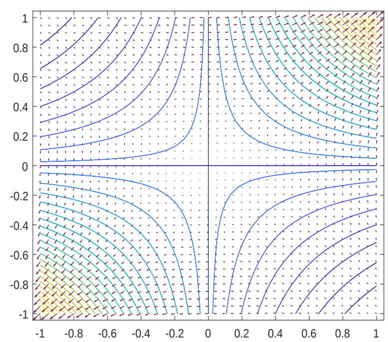
the contour lines are parabolas $x = y^2 + c$. The $x-$ axis is the symmetry axis.

d) $f_4 : \exp(x \cdot y)$

For $x = 0$ or $y = 0$ we have $f(x, y) = C = 1$. Hence the $x-$ axis and the $y-$ axis are contour lines.

The other contour lines are hyperbola branches $y = C/x$ for $x \neq 0$ or $x = C/y$ for $y \neq 0$.





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