Analysis III: Auditorium exercise class

Taylor Theorem, Lagrange-remainder,
Extrema of Multivariable Functions
Implicit Function Theorem

Sofiya Onyshkevych

November 22, 2021

BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

- VOLLSTÄNDIG GEIMPFTE
- GENESENE
- GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

- FULLY VACCINATED
- RECOVERED
- TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

Taylor Theorem

Let $D \subset \mathbb{R}^n$ be open and convex. Let $f : D \to \mathbb{R}$ be a C^{m+1}–function and $x_0 \in D$. Then the Taylor–expansion in $x \in D$ is well-defined

$$f(x) = T_m(x; x_0) + R_m(x; x_0),$$

where

$$R_m(x; x_0) = \sum_{|\alpha|=m+1} \frac{D^\alpha f(x_0 + \theta(x - x_0))}{\alpha!} (x - x_0)^\alpha, \forall \theta \in (0, 1)$$

is a Lagrange–remainder.

Error of a Taylor polynomial approximation

Examples

$R_2((x, y); (x_0, y_0)) =$

$R_3((x, y); (x_0, y_0)) =$

Exercise 1

Compute Taylor polynomial $T_2(x; x_0)$ of the function $f(x, y, z) = xe^z - y^2$ centered around a point $x_0 = (1, -1, 0)^T$.

Exercise 2

Compute the remainder in a Lagrange form $R_2(x; x_0)$ of $f(x, y, z) = xe^z - y^2$ centered around a point $x_0 = (1, -1, 0)^T$.

The estimate on the remainder

The Taylor approximation of a function reads as

$$f(x) = T_m(x; x_0) + O(\| x - x_0 \|^{m+1})$$

If $D^\alpha f, |\alpha| = m+1$ are bounded by $C > 0$ in a neighborhood of x_0 then the estimate holds

$$|R_m(x_0; x)| \leq \frac{n^{m+1}}{(m+1)!} C \| x - x_0 \|_\infty^{m+1}$$

Exercise 3

Compute $T_2(x; x_0)$ of a function $f(x, y) = e^x \cos(y)$ at the point $x_0 = (0, 0)^T$ and the estimate for the associated remainder $R_2(x; x_0)$ for $(x, y) \in [-2, 2] \times [-2, 2]$.

Exercise 4

Compute $T_2(x; x_0)$ of a function $f(x, y) = \cos(x^2 + y^2)$ at the point $x_0 = (0, 0)^T$ and the approximation error for $(x, y) \in [0, \frac{\pi}{4}] \times [0, \frac{\pi}{4}]$.

Extrema of multivariable function

Let $D \subset \mathbb{R}^n, f : D \to \mathbb{R}$ and $x_0 \in D$. Then at x_0 the function f has

- a (strict) global maximum if $\forall x \in D : f(x) \overset{(<)}{\leq} f(x_0)$
- a (strict) local maximum if

$$\exists \epsilon > 0 \; \forall x \in D \text{ with } \parallel x - x_0 \parallel < \epsilon : f(x) \overset{(<)}{\leq} f(x_0)$$

- analogously for minima

Note: x_0 is called an extremum if it is maximum or minimum

Stationary points

- The points $x_0 \in D$ for which it holds

$$\operatorname{grad} f(x_0) = 0$$

  are called stationary points (critical points) of f.
- Stationary points are not necessarily extrema.

Necessary optimality conditions

- Let $f \in D$ is $C^1, x_0 \in D$ - local extremum $\implies$ grad $f(x_0) = 0$
- Let $f \in D$ is $C^2, x_0 \in D$ - stationary point
    - if x_0 is local min (max) $\implies H\, f(x_0)$ positive (negative) semi-definite.

Sufficient optimality conditions

- Let $f \in D$ is $C^2, x_0 \in D$ - stationary point
  - $H f(x_0)$ positive (negative) definite $\implies x_0$ is strict local min (max)
  - $H f(x_0)$ indefinite $\implies x_0$ is a saddle point

Examples

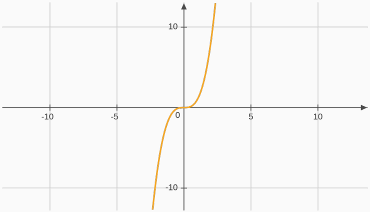

Figure 1: $f(x) = x^3, f'(0) = 0$ but $x^* = 0$ isn't extremum

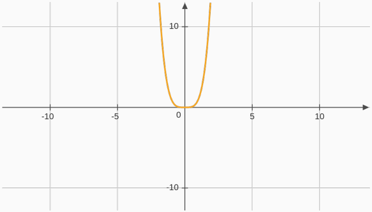

Figure 2: $f(x) = x^4, f''(x) = 12x^2, f''(0) = 0$ but still $x^* = 0$ is minimum

Exercise 5

Compute the stationary points of the following functions and determine whether it is min/max/saddle point

1. $f(x, y) = xy + x - 2y - 2$
2. $f(x, y) = 2x^3 - 3xy + 2y^3 - 3$
3. $f(x, y) = (x^2 + 2y^2)e^{-x^2-y^2}$
4. $f(x, y) = x^5 - 3x^3 + y^2 + 15,$

Implicitly Defined Functionss

Consider a system of nonlinear equations

$$g(x) = 0,$$

with $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$, $m < n$, i.e more unknowns than equations. - underdetermined system of equations.

We want to solve such systems locally expressing some variables via other.

Implicit Function Theorem

Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 - function. Let $(x, y) \in D$, where $x \in \mathbb{R}^{n-m}, y \in \mathbb{R}^m$. Let $(x_0, y_0) \in D$ - solution to $g(x_0, y_0) = 0$. If the Jacobian matrix

$$\frac{\partial g}{\partial y}(x_0, y_0) := \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \dots & \frac{\partial g_1}{\partial y_m} \\ \dots & & \dots \\ \frac{\partial g_m}{\partial y_1} & \dots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}$$

is regular, then there exist neighbourhoods U of x_0, V of $y_0, U \times V \subset D$ and a uniquely determined continuous differentiable function $f : U \to V : f(x_0) = y_0$ and $g(x, f(x)) = 0$ for all $x \in U$ and

$$J\,f(x) = -\left(\frac{\partial g}{\partial y}(x, f(x))\right)^{-1}\left(\frac{\partial g}{\partial x}(x, f(x))\right)$$

Exercise 7

Can the equation $(x^2 + y^2 + 2z^2)^{\frac{1}{2}} = cos(z)$ be solved uniquely for y in terms of x, z near $(0, 1, 0)$? For z in terms of x and y?

$F(x, y, z) =$

$F(0, 1, 0) =$

$\frac{\partial F}{\partial y}(0, 1, 0) =$

$\frac{\partial F}{\partial z}(0, 1, 0) =$

Exercise 8

Consider the function $F(x,y,z,u,v) : \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$F(x,y,z,u,v) = \begin{pmatrix} xy^2 + xzu + yv^2 - 3 \\ u^3yz + 2xv - u^2v^2 - 2 \end{pmatrix}$$

Can we solve for u, v as functions of x, y, z near $(1,1,1,1,1)$?

Notice that F(1,1,1,1,1) = 0.

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} =$$

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} (1,1,1,1,1) =$$

Some more exercises

Compute $T_2(x; x_0)$ of a function

$$f(x, y) = \cos(x)\sin(y)e^{x-y}$$

at the point $x_0 = (0, 0)^T$ and the associated remainder $R_2(x; x_0)$

Some more exercises

Compute Taylor polynomial of second degree $T_2(x; x_0)$ of a function

$$f(x, y) = \sin(x + y) + ye^{x-y}$$

at the point $x_0 = (0, 0)^T$ and the estimate for the Lagrange remainder for $|x| \leq 0.1, |y| \leq 0.1$.

Thank you!