Analysis III: Auditorium exercise class

Line Integrals, Potentials, Green's Theorem, Gauss' Theorem, Surface Integrals

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Line integrals for vector functions

- For a continuous vector field $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$, and a piecewise C^1-curve $c : [a, b] \to D$ the line integral on f over c is given by

$$\int_c f(x)\, dx := \int_a^b \langle f(c(t)), \dot{c}(t) \rangle \, dt$$

- For a closed curve $c(t)$, i.e. $c(a) = c(b)$, we use the notation

$$\oint_c f(x)\, dx$$

Exercise 1

Compute

$$\int_c f(x)\,dx,$$

where $f(x,y,z) = \begin{pmatrix} e^{2x} \\ z(y+1) \\ z^3 \end{pmatrix}$, $\quad c : [0,2] \to \mathbb{R}^3, \quad c(t) = \begin{pmatrix} t^3 \\ 1-3t \\ e^t \end{pmatrix}$.

Solution

We are given the parametrization of the curve, so we can start immediately with integration.

- The vector field along the curve:

$$f(c(t)) = \begin{pmatrix} e^{2t^3} \\ e^t(1-3t+1) \\ (e^t)^3 \end{pmatrix} = \begin{pmatrix} e^{2t^3} \\ e^t(2-3t) \\ e^{3t} \end{pmatrix}$$

The derivative of the parametrization:

$$\dot{c}(t) = \begin{pmatrix} 3t^2 \\ -3 \\ e^t \end{pmatrix}$$

The dot product:

$$\langle f(c(t)), \dot{c}(t)\rangle = e^{2t^3}\cdot 3t^2 + (-3)e^t(2-3t) + e^t\cdot e^{3t}$$

$$= 3t^2 e^{2t^3} - 3e^t(2-3t) + e^{4t}$$

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du$$

integration by parts

Now we evaluate the integral

$$\int_c f(x)dx = \int_0^2 \left(3t^2e^{2t^3} - 3e^t(2-3t) + e^{4t}\right)dt$$

$$= 3\cdot\frac{1}{3}\int_0^2 e^{2t^3}dt^3 - 6\int_0^2 e^t dt + 9\int_0^2 \underbrace{t}_{u}\,\underbrace{e^t dt}_{dv} + \frac{1}{4}\int_0^2 e^{4t}d(4t)$$

$$= \frac{1}{2}\int_0^2 e^{2t^3}d(2t^3) - 6e^t\Big|_0^2 + 9te^t\Big|_0^2 - 9\int_0^2 e^t dt + \frac{1}{4}e^{4t}\Big|_0^2$$

$$= \left(\frac{1}{2}e^{2t^3} - 3e^t(5-3t) + \frac{1}{4}e^{4t}\right)\Bigg|_0^2$$

$$= \frac{1}{2}e^{16} - \frac{1}{2} - 3e^2(5-6) + \underbrace{3(5-3\cdot 0)}_{15} + \frac{1}{4}e^8 - \frac{1}{4}$$

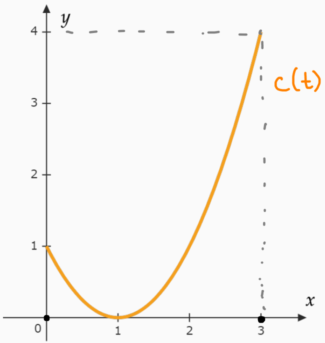
$$= \frac{1}{2}e^{16} + \frac{1}{4}e^8 + 3e^2 - \frac{57}{4}$$

Exercise 2

Compute

$$\int_c f(x)\,dx,$$

where $f(x,y) = \begin{pmatrix} y^2 \\ x^2 - 4 \end{pmatrix}$, and c is the portion of $y = (x-1)^2$ from $x = 0$ to $x = 3$.



Solution

We start by parametrizing the curve:

$c(t) = (t, (t-1)^2)^T \quad 0 \le t \le 3$

$c : [0,3] \longrightarrow \mathbb{R}^2$

Vector field along c:

$$f(c(t)) = \begin{pmatrix} (t-1)^4 \\ t^2-4 \end{pmatrix} \qquad \dot{c}(t) = \begin{pmatrix} 1 \\ 2(t-1) \end{pmatrix}$$

$$\langle f(c(t)), \dot{c}(t) \rangle = (t-1)^4 + 2(t-1)(t^2-4) = 2t^3 - 2t^2 - 8t + 8 + (t-1)^4$$

$$\int_c f(x)\,dx = \int_0^3 \left((t-1)^4 + 2t^3 - 2t^2 - 8t + 8\right) dt = \left(\frac{(t-1)^5}{5} + \frac{\cancel{2}\,t^4}{\cancel{4}_2} - 2\frac{t^3}{3} - \frac{8t^2}{2} + 8t\right)\Big|_0^3$$

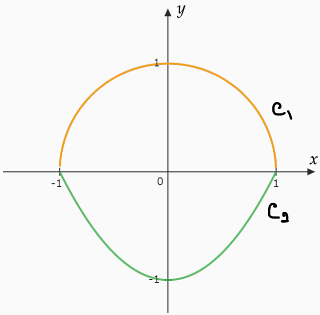
$$= \frac{(3-1)^5}{5} - \frac{(-1)^5}{5} + \frac{3^4}{2} - \frac{8\cdot 9}{2} + 24 = \ldots = \frac{171}{10}$$

Exercise 3

Compute

$$\int_c f(x)\,dx,$$

where $f(x,y) = \begin{pmatrix} 3y \\ x^2 - y \end{pmatrix}$, and c is the upper half unit-circle calculated at origin and the portion of $y = x^2 - 1$ from $x = -1$ to $x = 1$.



We have two curves to parametrize:

$c_1(t) = (\cos t, \sin t) \quad 0 \leq t \leq \pi$

$c_2(t) = (t, t^2 - 1) \quad -1 \leq t \leq 1$

Now compute the line integral for each of the curves:

$$f(c_1(t)) = \begin{pmatrix} 3\sin t \\ \cos^2 t - \sin t \end{pmatrix} \qquad \dot c_1(t) = \begin{pmatrix} -\sin t \\ \cos t \end{pmatrix}$$

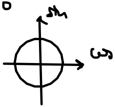
$$\int_{c_1} f(x)dx = \int_0^\pi \langle f(c_1(t)), \dot c_1(t)\rangle dt = \int_0^\pi \Big(-3\underbrace{\sin^2 t}_{=\frac{1-\cos(2t)}{2}} + \cos^3 t - \underbrace{\cos t \sin t}_{1/2\sin(2t)}\Big) dt$$

$$= \int_0^\pi \left(-\frac{3}{2}(1-\cos(2t)) + \cos t\,(1-\sin^2 t) - \frac{1}{2}\sin 2t\right) dt$$

$$= -\frac{3}{2}t\Big|_0^\pi + \frac{3}{2}\cdot\frac{1}{2}\int_0^\pi \cos(2t)\,d(2t) + \int_0^\pi \cos t\,dt - \int_0^\pi \sin^2 t\,d(\sin t) - \frac{1}{2}\cdot\frac{1}{2}\int_0^\pi \sin(2t)\,d(2t)$$

$$= -\frac{3}{2}\pi + \frac{3}{4}\sin(2t)\Big|_0^\pi + \sin t\Big|_0^\pi - \frac{\sin^3 t}{3}\Big|_0^\pi + \frac{1}{4}\cos(2t)\Big|_0^\pi$$

$$= -\frac{3}{2}\pi + \frac{3}{4}(0-0) + 0 - 0 + \frac{1}{4} - \frac{1}{4} = -\frac{3}{2}\pi$$



Analogously for c_2:

$$\int_{c_2} f(x)dx = \int_{-1}^{1}\left\langle \begin{pmatrix} 3(t^2-1) \\ t^2-(t^2-1) \end{pmatrix}, \begin{pmatrix} 1 \\ 2t \end{pmatrix}\right\rangle dt = \int_{-1}^{1}\big(3(t^2-1)+2t\big)dt = \left(3\frac{t^3}{3} - 2\frac{t^2}{2} - 3t\right)\Big|_{-1}^{1} = -4$$

Hence we have

$$\int_C f(x)\,dx = -\frac{3}{2}\pi + (-4).$$

Potentials

- Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. If there exists a scalar C^1-function $\varphi : D \to \mathbb{R}$ with

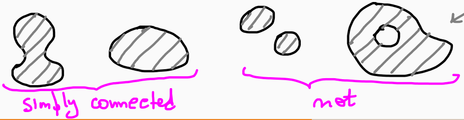
$$f(x) = \text{grad}\ \varphi(x)$$

  it is called a potential of $f(x)$.
- If there exists a potential for a vector field f, it is called a conservative field.
- Necessary condition for existence of a potential:

$$\text{curl}\ f(x) = 0 \quad \forall x \in D \tag{1}$$

'no holes'

- If D is simply connected, Eq. (1) is a sufficient condition.



connected but not simply

$\mathbb{R}^n$ is simply connected

The fundamental theorem of line integrals

For the continuous vector field $f : D \to \mathbb{R}^n$ with potential φ and a piecewise C^1-curve $c : [a, b] \to D$ it holds

$$\int_c f(x)\,dx = \varphi(c(b)) - \varphi(c(a))$$

Let $x_0 \in D$ - fixed point and c_x is an arbitrary piecewise C^1-curve in D connecting points x_0 and x. Then $\varphi(x)$ is given by

$$\varphi(x) = \int_{c_x} f(y)\,dy + \text{const}$$

Exercise 4

simply connected!

Let f be the vector field given by $f : \mathbb{R}^3 \to \mathbb{R}^3$:

$$f(x,y,y) = \begin{pmatrix} 3x^2y^4z^5+1 \\ 4x^3y^3z^5+2y \\ 5x^3y^4z^4+3z^2 \end{pmatrix} \begin{matrix} = f_1 \\ = f_2 \\ = f_3 \end{matrix}$$

- Show that there is a potential for f (without calculating it).
- Compute the potential using the definition (by integration)
- Compute the potential using the Fundamental theorem

- $\mathbb{R}^3$ is simply connected $\Rightarrow$ "curl $f = 0$" is a sufficient cond.

$$\operatorname{curl} f(x) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)i - \left(\frac{\partial f_3}{\partial x} - \frac{\partial f_1}{\partial z}\right)j + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)k$$

$$\operatorname{curl} f(x,y,z) = \begin{pmatrix} \frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z} \\ \frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x} \\ \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} 5x^3z^4 \cdot 4y^3 - 4x^3y^3 \cdot 5z^4 \\ 3x^2y^4 \cdot 5z^4 - 15x^2y^4z^4 \\ 12x^2y^3z^5 - 12x^2y^3z^5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow$ The condition (1) is satisfied $\Rightarrow$ there is a potential for f.

- Computing the potential using the integration. From def. of potential:

$$\begin{cases} \text{I} & \varphi_x(x,y,z) = f_1(x,y,z) \\ \text{II} & \varphi_y(x,y,z) = f_2(x,y,z) \\ \text{III} & \varphi_z(x,y,z) = f_3(x,y,z) \end{cases} \Longrightarrow \varphi_x(x,y,z) = 3x^2y^4z^5 + 1$$

$$\Rightarrow \varphi(x,y,z) = \int (3x^2y^4z^5 + 1)\,dx = x^3y^4z^5 + x + C(y,z) \quad (2)$$

$$\varphi_y(x,y,z) \overset{(2)}{=} \underline{x^3 \cdot 4y^3 \cdot z^5 + C_y(y,z) \overset{(\text{II})}{=} 4x^3y^3z^5 + 2y}$$

$$\Rightarrow C_y(y,z) = 2y \Rightarrow C(y,z) = \int 2y\,dy = y^2 + C(z)$$

plugging into (2): $\varphi(x,y,z) = x^3y^4z^5 + x + y^2 + C(z) \quad (3)$

$$\varphi_z(x,y,z) \overset{(3)}{=} \underline{x^3y^4 \cdot 5z^4 + C_z(z) \overset{(\text{III})}{=} 5x^3y^4z^4 + 3z^2}$$

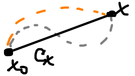
$$\Rightarrow C_z(z) = 3z^2 \Rightarrow C(z) = \int 3z^2\,dz = z^3 + k$$

plugging into (3) obtain the final formula for the potential of f:

$$\varphi(x,y,z) = x^3y^4z^5 + x + y^2 + z^3 + k.$$

- Computing the potential using the Fundamental Theorem:

choose a point x_0 and an arbitrary curve connecting x_0 to $x = (x_1, x_2, x_3)^T$.



Let $x_0 = (0,0,0)^T \in D$

Choose a curve: the simplest one – a line

Parametrize it: $c_x(t) = (x_1 t, x_2 t, x_3 t)$, $t \in [0,1]$

From Fundamental Theorem: $\varphi(x) = \int_{C_x} f(x)\,dx + k$

$\dot{c}_x(t) = (x_1, x_2, x_3)$

$$\int_{C_x} f(x)\,dx = \int_0^1 \langle f(c_x(t)), \dot{c}_x(t) \rangle\, dt$$

$$= \int_0^1 \left\langle \begin{pmatrix} 3x_1^2 x_2^4 x_3^5 t^{11} + 1 \\ 4x_1^3 x_2^3 x_3^5 t^{11} + 2x_2 t \\ 5x_1^3 x_2^4 x_3^4 t^{11} + 3x_3^2 t^2 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \right\rangle dt$$

$$= \int_0^1 \left(3x_1^3 x_2^4 x_3^5 t^{11} + x_1 + 4x_1^3 x_2^4 x_3^5 t^{11} + 2x_2^2 t + 5x_1^3 x_2^4 x_3^5 t^{11} + 3x_3^3 t^2 \right) dt$$

$$= 12 \int_0^1 x_1^3 x_2^4 x_3^5 t^{11}\, dt + \int_0^1 \left(x_1 + 2x_2^2 t + 3x_3^3 t^2 \right) dt$$

$$= \cancel{12}\, x_1^3 x_2^4 x_3^5 \frac{t^{12}}{\cancel{12}} \Big|_0^1 + \left(x_1 t + x_2^2 t^2 + x_3^3 t^3 \right) \Big|_0^1$$

$$= x_1^3 x_2^4 x_3^5 + x_1 + x_2^2 + x_3^3.$$

Let now $x_1 = x$ $x_2 = y$ $x_3 = z$.

$$\Rightarrow \varphi(x) = x^3 y^4 z^5 + x + y^2 + z^3 + k.$$

Green's Theorem

Let $f(x)$ be a C^1–vector field on a domain $D \subset \mathbb{R}^2$. Let $K \subset D$ be compact and projectable with respect to both coordinates, such that K is bounded by a closed and piecewise C^1–curve $c(t)$.

The parameterization of $c(t)$ is chosen such that K is always on the left when going along the curve with increasing parameter (positive circulation). Then the following holds

$$\oint_c f(x)\,dx = \int_K \operatorname{curl} f(x)\,dx$$

Exercise 5

Let

$$f(x, y) = (\sin x, \cos y)$$

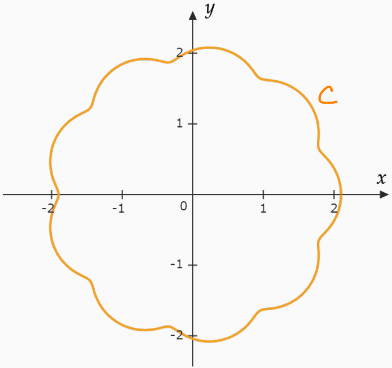
and let R be the region enclosed by the curve c parameterized by

$$c(t) = \begin{pmatrix} 2\cos t + \frac{1}{10}\cos(10t) \\ 2\sin t + \frac{1}{10}\sin(10t) \end{pmatrix}$$

on

$$0 \leq t \leq 2\pi.$$

Find the circulation around c.



Using Green's Theorem: $\oint_C f(x)dx = \iint_R \text{curl} f(x) dR$

$$\text{curl} f(x,y) = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \frac{\partial(\cos y)}{\partial x} - \frac{\partial(\sin x)}{\partial y} = 0 \quad \Rightarrow \oint_C f(x)dx = 0.$$

Alternatively: f is a conservative field, there is a potential φ: $f = \nabla\varphi$.

Let x^* - any point on C. Since C is closed x^* is beginning and end of C => using Fundamental Theorem:

$$\oint_C f dx = f(x^*) - f(x^*) = 0.$$

Exercise 6

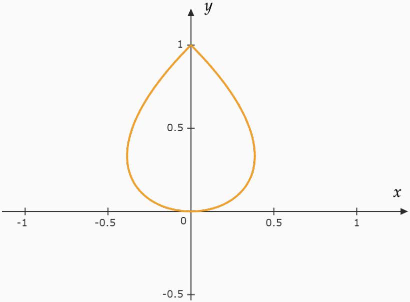
Let c be the closed curve parametrized by

$$c(t) = (t - t^3, t^2)$$

on

$$-1 \leq t \leq 1,$$

bounding the domain D. Compute the area of D using Green's theorem.



Green's theorem: $\oint_c f(x)\,dx = \iint_D \underbrace{\text{curl } f(x)}\, dD$

Area of $D = \iint_D 1 \cdot dD \Rightarrow$ choose a field such that curl $f(x) = 1$

The simplest one e.g. $f = \begin{pmatrix} -y \\ 0 \end{pmatrix}$

Check: $\text{curl } f(x,y) = \frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y} = \frac{\partial(0)}{\partial x} - \frac{\partial(-y)}{\partial y} = 1$ ✓

Now we have

$$\text{Area of } D = \iint_D 1 \cdot dD = \iint_D \text{curl } f(x)\, dD \overset{\text{Green's th.}}{=} \oint_c f(x)\, dx \qquad \dot{c}(t) = \begin{pmatrix} 1-3t^2 \\ 2t \end{pmatrix}$$

$$= \int_{-1}^{1} \langle (-t^2, 0), (1-3t^2, 2t) \rangle \, dt$$

$$= \int_{-1}^{1} (-t^2 + 3t^4)\, dt = \left(-\frac{t^3}{3} + \frac{3t^5}{5} \right) \Bigg|_{-1}^{1}$$

$$= -\frac{1}{3} - \frac{1}{3} + \frac{3}{5} + \frac{3}{5} = -\frac{2}{3} + \frac{6}{5} = \frac{8}{15}$$

Surface integrals

r is a parametrization

- Scalar surface integral ($f : D \to \mathbb{R}$)

$$\iint_S f(x,y,z)\, dS = \iint_{\underbrace{r(S)}} f \cdot dS = \iint_K f(r(u,v)) \cdot \underbrace{\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|}\, dudv$$

- Flux integral (of a vector field $f : D \to \mathbb{R}^3$)

$$\iint_S f \cdot n\, dS = \iint_{r(S)} f \cdot dS = \iint_K f(r(u,v)) \cdot \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\|\, dudv$$

$$= \iint_K \langle f(r(u,v)), \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \rangle\, dudv$$

- Gauss Theorem (Divergence theorem)

$$\int_G \operatorname{div} f(x)\, dx = \int_{\partial G} \langle f(x), \underbrace{n(x)}_{\text{outer normal}} \rangle\, dS = \int_K \langle f(r(u,v)), \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \rangle\, dudv$$

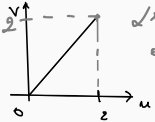
Exercise 7

scalar ⇒ 1st formula

Calculate the surface integral $\iint_S 5 dS$, where S is the surface with parametrization

$$r(u, v) = (u, u^2, v)$$

for $0 \le u \le 2$ and $0 \le v \le u$.



projection on uv

Using the formula: $\iint_S f(x,y,z)\,dS = \iint_K f(r(u,v)) \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\,dv$

$\frac{\partial r}{\partial u} = (1, 2u, 0)$; $\frac{\partial r}{\partial v} = (0, 0, 1)$

First compute the cross product:

$$\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} = \begin{vmatrix} i & j & k \\ 1 & 2u & 0 \\ 0 & 0 & 1 \end{vmatrix} = (2u, -1, 0)$$

$$\left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| = \sqrt{4u^2+1}$$

$$\iint_S 5\,dS = 5\iint_K \sqrt{4u^2+1}\,dK = 5\int_0^2\int_0^4 \sqrt{1+4u^2}\,dv\,du = 5\int_0^2 \sqrt{1+4u^2}\cdot v\Big|_0^4\,du$$

$$= 5\int_0^2 \sqrt{1+4u^2}\,4\,du = \frac{5}{2}\int_0^2 \sqrt{1+4u^2}\cdot\frac{1}{4}\,d(4u^2+1) = \frac{5}{8}\,\frac{1}{\frac{1}{2}+1}\left(1+4u^2\right)^{3/2}\Big|_0^2 = \dots$$

Exercise 7

Let

$$v(x, y, z) = (2x, 2y, z)$$

be the velocity field of a fluid with constant density $\rho = 8$. Let S be given by

$$x^2 + y^2 + z^2 = 9 \text{ and } z \geq 0$$

$r = 3$

such that S is oriented outward.

Compute the mass flow rate of the fluid across S.

$\iint_S \rho v \cdot dS$ - ?

start by parametrizing the surface:

$$r(\varphi, \Theta) = \begin{pmatrix} 3\cos\varphi\cos\Theta \\ 3\sin\varphi\cos\Theta \\ 3\sin\Theta \end{pmatrix} \qquad \Theta \in [0; \tfrac{\pi}{2}] \qquad \varphi \in [0; 2\pi]$$

$$\frac{\partial r}{\partial \varphi} \times \frac{\partial r}{\partial \theta} = \begin{vmatrix} i & j & k \\ -3\sin\varphi\cos\theta & 3\cos\varphi\cos\theta & 0 \\ -3\cos\varphi\sin\theta & -3\sin\varphi\sin\theta & 3\cos\theta \end{vmatrix}$$

$$= \begin{pmatrix} 9\cos^2\theta\cos\varphi \\ +9\cos^2\theta\sin\varphi \\ \underbrace{9\sin^2\varphi\sin\theta\cos\theta + 9\cos^2\varphi\cos\theta\sin\theta}_{9\cos\theta\sin\theta\underbrace{(\sin^2\varphi+\cos^2\varphi)}_{1}} \end{pmatrix}$$

$\forall$ components are positive $\Rightarrow$ vector is outward.

$$\iint_S \rho\, v \cdot dS = 8 \iint_K v(r(\varphi,\theta)) \cdot \left(\frac{\partial r}{\partial \varphi} \times \frac{\partial r}{\partial \theta}\right) d\theta\, d\varphi$$

$$= 8 \int_0^{2\pi} \int_0^{\pi/2} \begin{pmatrix} 6\cos\varphi\cos\theta \\ 6\sin\varphi\cos\theta \\ 3\sin\theta \end{pmatrix} \begin{pmatrix} 9\cos^2\theta\cos\varphi \\ 9\cos^2\theta\sin\varphi \\ 9\cos\theta\sin\theta \end{pmatrix} d\theta\, d\varphi$$

$$= 8\int_0^{2\pi}\int_0^{\pi/2}\left(54\left(\underbrace{\cos^2\varphi\cos^3\theta+\cos^3\theta\,\sin^2\varphi}\right)+27\sin^2\theta\cos\theta\right)d\theta\,d\varphi$$

$$= 8\int_0^{2\pi}\int_0^{\pi/2} 54\cos^3\theta + 27\sin^2\theta\cos\theta\,d\theta\,d\varphi$$

$$= 8\int_0^{2\pi}\int_0^{\pi/2} 54\quad(1-\sin^2\theta)+\quad 27\sin^2\theta\,d(\sin\theta)\,d\varphi$$

$$= 8\int_0^{2\pi}\int_0^{\pi/2} 54 - 27\sin^2\theta\,d\sin\theta\,d\varphi$$

$$= 8\int_0^{2\pi}\left(54\underbrace{\sin\theta}_{1} - 9\underbrace{\sin^3\theta}_{1}\right)\Big|_0^{\pi/2}d\varphi$$

$$= 8\cdot 45\int_0^{2\pi}d\varphi = 8\cdot 45\,\varphi\Big|_0^{2\pi} = 720\pi$$

Thank you!