# Analysis III: Auditorium exercise class

## Line Integrals, Potentials, Green’s Theorem, Gauss’ Theorem, Surface Integrals

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# Line integrals for vector functions

- For a continuous vector field $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$, and a piecewise $C^1$-curve $c : [a, b] \to D$ the line integral on $f$ over $c$ is given by

$$\int_c f(x)\, dx := \int_a^b \langle f(c(t)), \dot{c}(t) \rangle \, dt$$

- For a closed curve $c(t)$, i.e. $c(a) = c(b)$, we use the notation

$$\oint_c f(x)\, dx$$

# Exercise 1

Compute

$$\int_c f(x)\,dx,$$

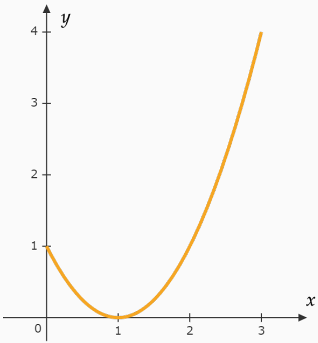
where $f(x,y,z) = \begin{pmatrix} e^{2x} \\ z(y+1) \\ z^3 \end{pmatrix}, \quad c : [0,2] \to \mathbb{R}^3, \quad c(t) = \begin{pmatrix} t^3 \\ 1-3t \\ e^t \end{pmatrix}.$

# Exercise 2

Compute

$$\int_c f(x)\,dx,$$

where $f(x,y) = \begin{pmatrix} y^2 \\ x^2 - 4 \end{pmatrix}$, and $c$ is the portion of $y = (x-1)^2$ from $x = 0$ to $x = 3$.

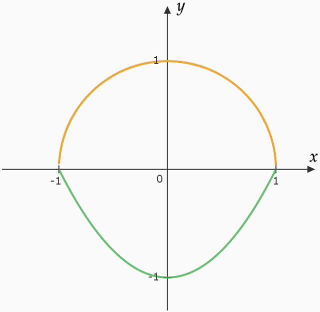


# Exercise 3

Compute

$$\int_c f(x)\,dx,$$

where $f(x, y) = \begin{pmatrix} 3y \\ x^2 - y \end{pmatrix}$, and $c$ is the upper half unit-circle calculated at origin and the portion of $y = x^2 - 1$ from $x = -1$ to $x = 1$.



# Potentials

- Let $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ be a vector field. If there exists a scalar $C^1$-function $\varphi : D \to \mathbb{R}$ with

$$f(x) = \text{grad } \varphi(x)$$

  it is called a potential of $f(x)$.
- If there exists a potential for a vector field $f$, it is called a conservative field.
- Necessary condition for existence of a potential:

$$\text{curl } f(x) = 0 \quad \forall x \in D \tag{1}$$

- If $D$ is simply connected, Eq. (1) is a sufficient condition.

# The fundamental theorem of line integrals

For the continuous vector field $f : D \to \mathbb{R}^n$ with potential $\varphi$ and a piecewise $C^1$-curve $c : [a, b] \to D$ it holds

$$\int\limits_c f(x)\, dx = \varphi(c(b)) - \varphi(c(a))$$

Let $x_0 \in D$ - fixed point and $c_x$ is an arbitrary piecewise $C^1$-curve in $D$ connecting points $x_0$ and $x$. Then $\varphi(x)$ is given by

$$\varphi(x) = \int\limits_{c_x} f(y)\, dy + \text{const}$$

# Exercise 4

Let $f$ be the vector field given by

$$f(x, y, y) = \begin{pmatrix} 3x^2y^4z^5 + 1 \\ 4x^3y^3z^5 + 2y \\ 5x^3y^4z^4 + 3z^2 \end{pmatrix}.$$

- Show that there is a potential for $f$ (without calculating it).
- Compute the potential using the definition (by integration)
- Compute the potential using the Fundamental theorem

# Green's Theorem

Let $f(x)$ be a $C^1$–vector field on a domain $D \subset \mathbb{R}^2$. Let $K \subset D$ be compact and projectable with respect to both coordinates, such that $K$ is bounded by a closed and piecewise $C^1$–curve $c(t)$.

The parameterization of $c(t)$ is chosen such that $K$ is always on the left when going along the curve with increasing parameter (positive circulation). Then the following holds

$$\oint\limits_{c} f(x)\,dx = \int\limits_{K} \operatorname{curl} f(x)\,dx$$

# Exercise 5

Let

$$f(x, y) = (\sin x, \cos y)$$

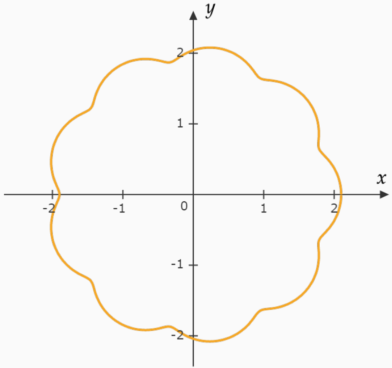
and let $R$ be the region enclosed by the curve $c$ parameterized by

$$c(t) = \begin{pmatrix} 2\cos t + \frac{1}{10}\cos(10t) \\ 2\sin t + \frac{1}{10}\sin(10t) \end{pmatrix}$$

on

$$0 \leq t \leq 2\pi.$$

Find the circulation around $c$.



# Exercise 6

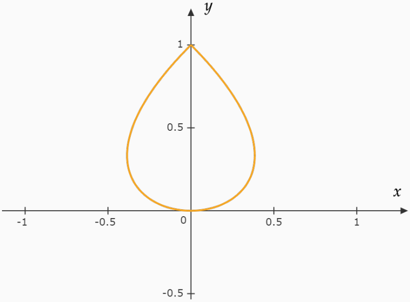
Let $c$ be the closed curve parametrized by

$$c(t) = (t - t^3, t^2)$$

on

$$-1 \leq t \leq 1,$$

bounding the domain $D$. Compute the area of $D$ using Green's theorem.



# Surface integrals

- Scalar surface integral ($f : D \to \mathbb{R}$)

$$\iint_S f(x,y,z)\, dS = \iint_{r(S)} f \cdot dS = \iint_K f(r(u,v)) \cdot \parallel \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \parallel\, dudv$$

- Flux integral (of a vector field $f : D \to \mathbb{R}^3$)

$$\begin{aligned}\iint_S f \cdot n\, dS &= \iint_{r(S)} f \cdot dS = \iint_K f(r(u,v)) \cdot \parallel \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \parallel\, dudv \\ &= \iint_K \langle f(r(u,v)), \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \rangle\, dudv\end{aligned}$$

- Gauss Theorem (Divergence theorem)

$$\int_G \operatorname{div} f(x)\, dx = \int_{\partial G} \langle f(x), n(x) \rangle\, dS = \int_K \langle f(r(u,v)), \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \rangle\, dudv$$

# Exercise 7

Calculate the surface integral $\iint\limits_S 5dS$, where $S$ is the surface with parametrization

$$r(u, v) = (u, u^2, v)$$

for $0 \leq u \leq 2$ and $0 \leq v \leq u$.

# Exercise 7

Let

$$v(x, y, z) = (2x, 2y, z)$$

be the velocity field of a fluid with constant density $\rho = 8$. Let $S$ be given by

$$x^2 + y^2 + z^2 = 9 \text{ and } z \geq 0$$

such that $S$ is oriented outward.

Compute the mass flow rate of the fluid across $S$.

Thank you!