

Analysis III: Auditorium exercise class

Double integrals, Theorem of transfrmation,
Center of mass and moment of inertia,
Steiner's Theorem

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January 2, 2022

Fubini theorem:

- If f be integrable over D and for all $x \in [a_1, b_1], y \in [a_2, b_2]$ there exist integrals

$$F(x) = \int_{a_2}^{b_2} f(x, y) dy \quad G(y) = \int_{a_1}^{b_1} f(x, y) dx$$

then it holds

$$\int_D f(x) dx = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y) dy dx = \int_{a_2}^{b_2} \int_{a_1}^{b_1} f(x, y) dx dy$$

Previously: Computing area using double integrals

Exercise 1

Determine the area of the region between the two curves $y = x^2 - 2$ and $y = 2$.

$$x^2 - 2 = 2 \Rightarrow x = \pm 2 - \text{points of intersection}$$

$$\Rightarrow -2 \leq x \leq 2$$

$$x^2 - 2 \leq y \leq 2$$

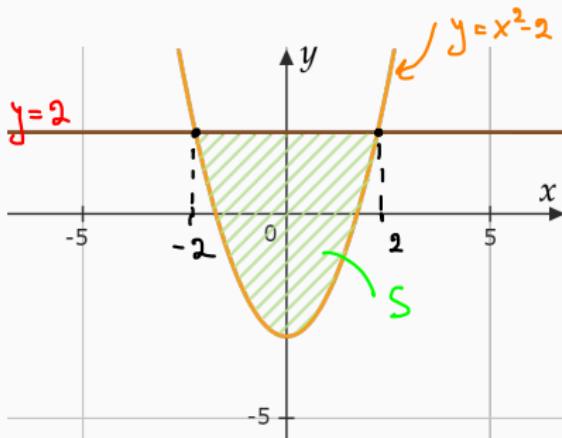
$$\Rightarrow S = \int_{-2}^2 \int_{x^2-2}^2 1 dy dx = \int_{-2}^2 \left(y \Big|_{x^2-2}^2 \right) dx$$

$$= \int_{-2}^2 (2 - (x^2 - 2)) dx = \left(4x - \frac{x^3}{3} \right) \Big|_{-2}^2 = \frac{32}{3}$$

Alternatively as difference of functions:

$$S = \int_{-2}^2 (x^2 - 2) - 2 dx = \frac{32}{3}$$

T careful with this if they intersect!



Previously: Computing area using double integrals

Exercise 2

Determine the area of the region between the two curves $y = x^3$ and $y = x^2 + x$.

$$x^2 + x = x^3$$

$$x=0$$

$$x^2 - x - 1 = 0$$

$$\Delta = 1+4=5$$

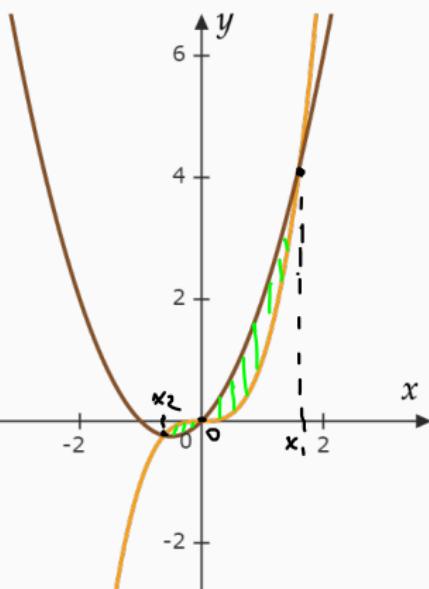
$$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

Separate into two integrals:

$$S = J_1 + J_2$$

$$J_1 = \int_{\frac{1-\sqrt{5}}{2}}^0 \int_{x^2+x}^{x^3} 1 \, dy \, dx = \int_{\frac{1-\sqrt{5}}{2}}^0 (x^3 - (x^2 + x)) \, dx$$

$$= \left(\frac{x^4}{4} - \frac{x^3}{3} - \frac{x^2}{2} \right) \Big|_{\frac{1-\sqrt{5}}{2}}^0 = \left(\frac{3-\sqrt{5}}{2} \right) \left(\frac{9-3\sqrt{5}+4-4\sqrt{5}-12}{24} \right) = \dots$$



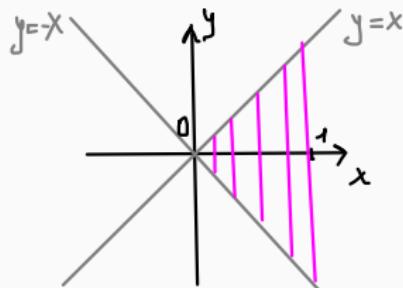
$$\begin{aligned}
 J_2 &= \int_0^{\frac{1+\sqrt{5}}{2}} \int_{x^3}^{x^2+x} 1 dy dx = \int_0^{\frac{1+\sqrt{5}}{2}} x^2 + x - x^3 dx = \left. \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{4} \right|_0^{\frac{1+\sqrt{5}}{2}} \\
 &= x^2 \left(\frac{x}{3} + \frac{1}{2} - \frac{x^2}{4} \right) \Big|_0^{\frac{1+\sqrt{5}}{2}} = \frac{3+\sqrt{5}}{2} \left(\frac{4+4\sqrt{5}+12-9-3\sqrt{5}}{24} \right) = \dots
 \end{aligned}$$

$$S = J_1 + J_2 =$$

Computing volume using double integrals

Exercise 3

Compute the volume of the body D defined by $0 < x < 1$, $-x < y < x$ and $0 < z < 3 + x^2 - 2y$.

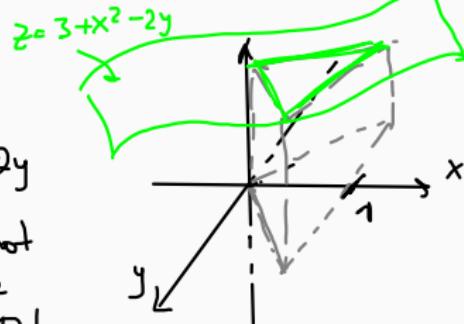


$$0 < x < 1$$

$$-x < y < x$$

$$0 < z < 3 + x^2 - 2y$$

this surface does not
intersect the xy plane
within the body D !



\Rightarrow do not have to write as two integrals

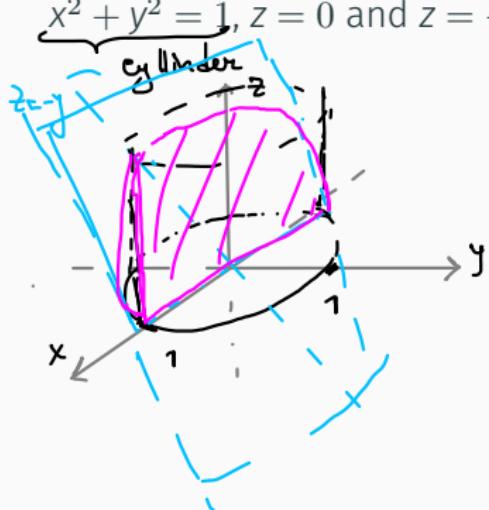
$$\begin{aligned} V &= \int_0^1 \int_{-x}^x \int_0^{3+x^2-2y} 1 dz dy dx = \int_0^1 \int_{-x}^x (3+x^2-2y) dy dx = \int_0^1 [3y + x^2 y - \frac{2y^2}{2}] \Big|_{-x}^x dx \\ &= \int_0^1 \left(3x + x^3 - x^2 - (-3x - x^3 - x^2) \right) dx = \int_0^1 (6x + 2x^3) dx = \left[\frac{6x^2}{2} + \frac{2x^4}{4} \right]_0^1 = \frac{6}{2} + \frac{2}{4} = \frac{14}{4} = \frac{7}{2} \end{aligned}$$

Computing volume using double integrals

Exercise 4 (skipped)

Compute the volume of space region D bounded by the surfaces

$$x^2 + y^2 = 1, z = 0 \text{ and } z = -y.$$



$$\begin{aligned} & \begin{array}{l} -1 \leq x \leq 1 \\ 0 \leq z \leq -y \\ -\sqrt{1-x^2} \leq y \leq 0 \end{array} \\ & \Rightarrow \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 \int_{-y}^{1} 1 dz dy dx \\ & = \int_{-1}^1 \int_{-\sqrt{1-x^2}}^0 (-y) dy dx = \int_{-1}^1 \left(-\frac{y^2}{2} \right) \Big|_{-\sqrt{1-x^2}}^0 dx \\ & = \int_{-1}^1 \frac{1-x^2}{2} dx = \left(\frac{1}{2}x - \frac{x^3}{6} \right) \Big|_{-1}^1 = \frac{2}{3} \end{aligned}$$

Coordinate Transformations

a. polar coordinates

$$0 \leq r \leq R, \quad 0 \leq \varphi \leq 2\pi$$

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \end{cases} \Phi(r, \varphi) \Rightarrow \det(J\Phi(r, \varphi)) = r$$

b. cylindrical coordinates

$$0 \leq r \leq R \quad 0 \leq \varphi \leq 2\pi \quad 0 < z < h$$

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ z = z \end{cases} \det(J\Phi(r, \varphi, z)) = r$$

c. spherical coordinates

$$\begin{cases} x = r \cos \varphi \cos \theta \\ y = r \sin \varphi \cos \theta \\ z = r \sin \theta \end{cases} \det(J\Phi(r, \varphi, \theta)) = \underline{\underline{r^2 \cos \theta}}$$

$$0 \leq r \leq R \quad 0 \leq \varphi \leq 2\pi \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Theorem of transformation

Let $U \subset \mathbb{R}^n$ be compact, measurable set, $\Phi : D \rightarrow \mathbb{R}^n$ be C^1 -coordinate transform. So the transformation Φ is invertible on D^0 . Let further $K = \Phi(D)$. Then for continuous function $f : K \subset \mathbb{R}^n \rightarrow \mathbb{R}$ it holds

$$\int_K f(\mathbf{x}) d\mathbf{x} = \int_D f(\Phi(\mathbf{u})) \cdot |\det(J\Phi(\mathbf{u}))| d\mathbf{u}$$

Computing integrals using coordinate transformations

Exercise 5

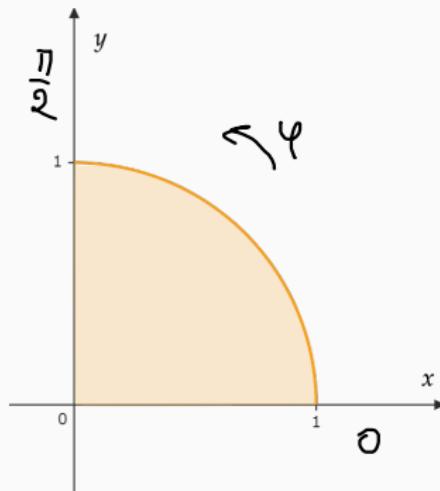
Compute the following integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2) dy dx$$

Solution: use polar coordinates

$$\begin{cases} x = r \cos \phi \\ y = r \sin \phi \end{cases} \quad \begin{matrix} 0 \leq \phi \leq \frac{\pi}{2} \\ 0 \leq r \leq 1 \end{matrix}$$

$$\begin{aligned} I &= \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \underbrace{\det(J\phi)}_{r} dr d\phi \\ &= \int_0^{\frac{\pi}{2}} \frac{r^4}{4} \Big|_0^1 d\phi = \frac{1}{4} \phi \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{8} \end{aligned}$$



Computing integrals using coordinate transformations

Exercise 6

Calculate the integral

$$\iiint_U e^{(x^2+y^2+z^2)^{\frac{3}{2}}} dx dy dz,$$

where the region U is the unit ball $x^2 + y^2 + z^2 \leq 1$.

$\text{L} = 1$

The ball is centered in origin \Rightarrow
using spherical coordinates

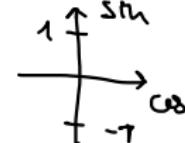
$$\begin{aligned} 0 &\leq r \leq 1 \\ 0 &\leq \varphi \leq 2\pi \\ -\frac{\pi}{2} &\leq \theta \leq \frac{\pi}{2} \end{aligned}$$

$$\begin{aligned} x &= r \cos \varphi \sin \theta \\ y &= r \sin \varphi \sin \theta \\ z &= r \sin \theta \end{aligned}$$

$$\begin{aligned} x^2 + y^2 + z^2 &= r^2 \cos^2 \varphi \cos^2 \theta + r^2 \sin^2 \varphi \sin^2 \theta + r^2 \sin^2 \theta \\ &= r^2 \cos^2 \theta (\underbrace{\cos^2 \varphi + \sin^2 \theta}_{1}) + r^2 \sin^2 \theta \\ &= r^2 (\cos^2 \theta + \sin^2 \theta)^2 = r^2 \end{aligned}$$

$$J = \iiint_{U'} e^{r^2 \frac{\theta^3}{3}} r^2 \cos \Theta dr d\varphi d\Theta = \int_0^{2\pi} d\varphi \int_0^1 e^{r^3} r^2 dr \int_{-\pi/2}^{\pi/2} \cos \Theta d\Theta$$

$$= \varphi \Big|_0^{2\pi} \cdot \left(e^{r^3} \frac{1}{3} dr^3 \right) \cdot \left(\sin \Theta \Big|_{-\pi/2}^{\pi/2} \right)$$



$$= 2\pi \cdot \frac{1}{3} e^{r^3} \Big|_0^1 \left(\underbrace{\sin \frac{\pi}{2}}_1 - \underbrace{\sin(-\frac{\pi}{2})}_{-1} \right) = \frac{2\pi}{3} (e-1) \cdot 2$$

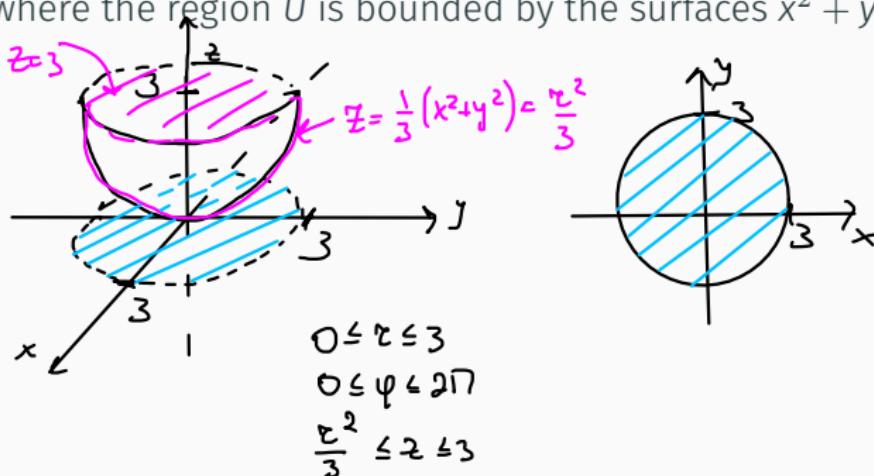
Computing integrals using coordinate transformations

Exercise 7

Calculate the integral

$$\iiint_U (x^2 + y^2) \, dx dy dz,$$

where the region U is bounded by the surfaces $x^2 + y^2 = 3z$, $z = 3$.



Use cylindrical
coord:

$$\begin{cases} x = r \cos \varphi \\ y = r \sin \varphi \\ z = z \end{cases}$$

$$\det(J\Phi(r, \varphi, z)) = r$$

$$\begin{aligned}
 J &= \iiint_U (x^2 + y^2) dx dy dz = \iiint_{U'} r^2 r dr d\varphi dz = \int_0^{2\pi} d\varphi \int_0^3 r^3 dr \int_{r^2/3}^3 dz \\
 &= \int_0^{2\pi} d\varphi \int_0^3 r^3 z \Big|_{r^2/3}^3 dr = \int_0^{2\pi} d\varphi \int_0^3 r^3 \left(3 - \frac{r^2}{3}\right) dr \\
 &= \int_0^{2\pi} d\varphi \int_0^3 \left(3r^3 - \frac{r^5}{3}\right) dr = \int_0^{2\pi} d\varphi \left(\frac{3r^4}{4} - \frac{r^6}{18}\right) \Big|_0^3 \\
 &= \frac{2\pi}{2} \left(\frac{3^5}{4} - \frac{3^6}{18}\right) = \frac{3^5 \pi}{2} \left(\frac{1}{2} - \frac{3}{14}\right) = \frac{1}{2} \frac{3^5 \pi}{18} \left(\frac{7-3}{3}\right) = \frac{3^4 \pi}{2} = \frac{81\pi}{2}
 \end{aligned}$$

Application

Let $K \subset \mathbb{R}^3$ be the body with the continuous *mass density function* $\rho : K \rightarrow \mathbb{R}$.

- The **mass** M of the body K is given by

$$M = \int_K \rho(x, y, z) dK \quad \text{iff } d(x, y, z) = dx dy dz$$

- The **center of mass** of K is computed as

$$\mathbf{x}_s = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \int_K \rho(x, y, z) \cdot x dK \\ \int_K \rho(x, y, z) \cdot y dK \\ \int_K \rho(x, y, z) \cdot z dK \end{pmatrix}$$

Computing the total mass

Exercise 8

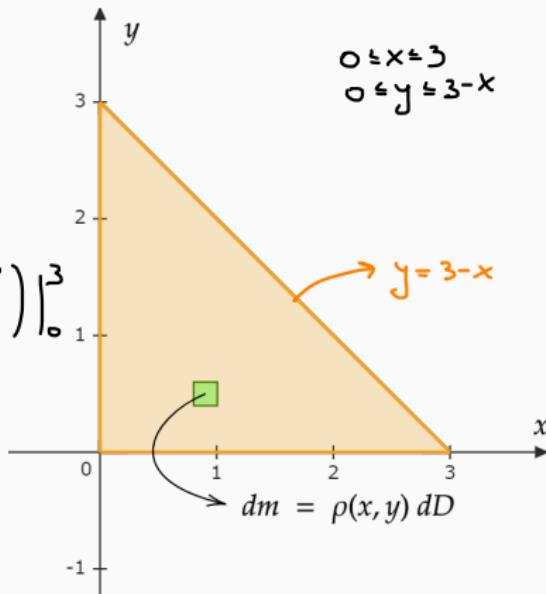
Consider a triangular domain D with vertices $(0, 0)$, $(0, 3)$, $(3, 0)$ and with density $\rho(x, y) = xy$. Find the total mass of D .

$$M = \iint_D dm = \iint_D \rho(x, y) dD$$

$$= \int_0^3 \int_0^{3-x} xy dy dx = \int_0^3 \left(\frac{xy^2}{2} \right) \Big|_0^{3-x} dx$$

$$= \int_0^3 \frac{x(9 - 6x + x^2)}{2} dx = \left(\frac{9x^2}{2} - \frac{6x^3}{8} + \frac{x^4}{8} \right) \Big|_0^3$$

$$= \dots = \frac{27}{8}$$



Computing the center of mass

Exercise 9

Consider the same triangular domain D with vertices $(0, 0)$, $(0, 3)$, $(3, 0)$ and with density $\rho(x, y) = xy$. Find the center of mass.

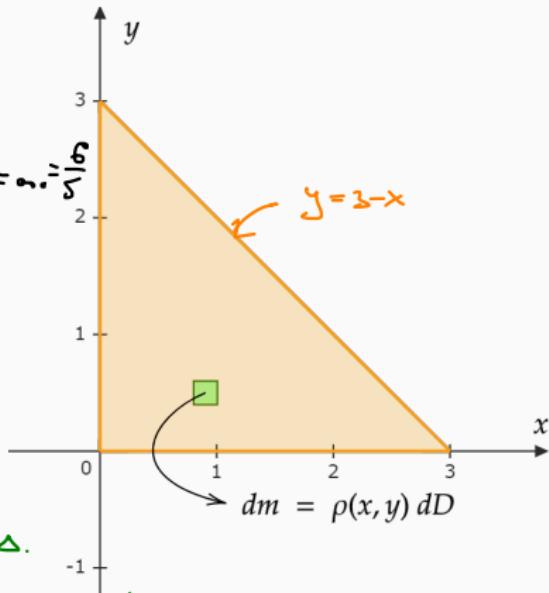
$$x_s = \frac{\iint_D x\rho(x, y) dD}{M} = \frac{8}{27} \iint_0^3 \int_0^{3-x} x^2 y dy dx$$
$$= \frac{8}{27} \int_0^3 x^2 \frac{(3-x)^2}{2} dx = \frac{4}{27} \left(\frac{9x^3}{3} - \frac{6x^4}{4} + \frac{x^5}{5} \right) \Big|_0^3 = \dots = \frac{6}{5}$$

$$y_s = \frac{\iint_D y\rho(x, y) dD}{M} = \frac{8}{27} \iint_0^3 \int_0^{3-x} xy^2 dy dx$$
$$= \dots = \frac{6}{5}$$

$x_s = \left(\frac{6}{5}, \frac{6}{5} \right)$ - not the same as centroid of Δ .

Centroid \Rightarrow if $f = \text{const}$ $x_{cc} = \frac{8}{27} \iint_0^3 \int_0^{3-x} dy dx = \dots = 1$

$y_{cc} = \dots = 1$

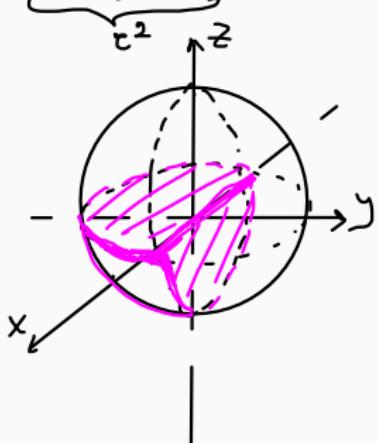


Computing the center of mass

Exercise 10

Sketch the quarter sphere K given by $y \leq 0, z \leq 0, x^2 + y^2 + z^2 \leq 16$ and compute its center of mass for the given density function

$\rho = x^2 + y^2 + z^2 + 1$ using ~~spherical, since in 3D~~ polar coordinates.



$$0 \leq r \leq 4 \quad \pi \leq \varphi \leq 2\pi \quad -\frac{\pi}{2} \leq \theta \leq 0$$

First compute the mass:

$$\begin{aligned} M &= \int_K x^2 + y^2 + z^2 + 1 \, dV = \int_{K'} (r^2 + 1) r^2 \sin \theta \, dr \, d\theta \, d\varphi \\ &= \int_0^4 \int_0^{2\pi} \int_{-\pi/2}^{\pi} (r^2 + 1) r^2 \sin \theta \, d\theta \, d\varphi \, dr \\ &= \int_0^4 \int_0^{2\pi} (r^4 + r^2) \left(\sin \theta \Big|_{-\pi/2}^{\pi} \right) \, d\varphi \, dr \end{aligned}$$

$$= \int_0^{\frac{\pi}{2}} \int_{\pi}^{2\pi} (r^4 + r^2) d\varphi dr = \int_0^{\frac{\pi}{2}} \left((r^4 + r^2) \varphi \Big|_{\pi}^{2\pi} \right) dr = \int_0^{\frac{\pi}{2}} (r^4 + r^2) dr$$

$$= \pi \left(\frac{r^5}{5} + \frac{r^3}{3} \right) \Big|_0^{\frac{\pi}{2}} = \pi \left(\frac{\pi^5}{5} + \frac{\pi^3}{3} \right) = \frac{\pi^3}{16.4} \left(\frac{16 \cdot 3 + 5}{15} \right) \pi = \frac{53.64}{15} \pi = \frac{733.32}{15}$$

Now computing center of mass.

$$x_s = \frac{1}{M} \int_K (x^2 + y^2 + z^2 + 1) x dV = \int_0^{\frac{\pi}{2}} \int_{\pi}^{2\pi} \int_{-\pi/2}^0 (r^4 + 1) r \cos \varphi \sin \theta r^2 \cos \theta d\theta d\varphi dr$$

$$2l(\frac{1}{2})$$

$$\text{if } \cos 2\theta = \frac{1 + \cos 2\theta}{2}$$

$$= \frac{1}{M} \int_0^{\frac{\pi}{2}} \int_{\pi}^{2\pi} \int_{-\pi/2}^0 (r^5 + r^3) \cos \varphi \frac{1 + \cos 2\theta}{4} d(2\theta) d\varphi dr$$

$$= \frac{1}{M} \int_0^{\frac{\pi}{2}} \int_{\pi}^{2\pi} (r^5 + r^3) \cos \varphi \left(\frac{2\theta + \sin 2\theta}{4} \Big|_{-\pi/2}^{\pi} \right) d\varphi dr$$

$$= \frac{\pi}{4M} \int_0^{\frac{\pi}{2}} (r^5 + r^3) \sin \varphi \Big|_{\pi}^{2\pi} dr = \frac{\pi}{4M} \int_0^{\frac{\pi}{2}} (r^5 + r^3) \cdot 0 = 0$$

$$y_s = \frac{1}{M} \int_0^{\frac{\pi}{2}} \int_{\pi}^{2\pi} \int_{-\pi/2}^0 (r^4 + 1) \underbrace{\cancel{r \sin \varphi \sin \theta r^2}}_{J(\theta)} \underbrace{\cos \theta}_{d\theta} d\varphi dr = \dots = -\frac{175}{106}$$

Analogously z_s .

Application

- The moment of inertia of K with respect to the given axis of rotation A is given by

→ exercise
class

$$\Theta_A = \int_K \rho(x, y, z) r^2(x, y, z) dK,$$

where $r(x, y, z)$ is the distance from the point $(x, y, z)^T \in K$ to the axis A .

Theorem (Steiner's Theorem)

Let S be the axis parallel to A going through the center of the mass \mathbf{x}_s of a body K , d the distance to the axis A from the point \mathbf{x}_s , M - mass of K , ρ - constant density of K . Then the moment of inertia w.r.t A is given by

$$\Theta_A = Md^2 + \Theta_S$$

Computing moment of inertia

Exercise 11

Consider the triangular region D with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$ and with density $\rho(x, y) = xy$. Find the moments of inertia w.r.t axes.

$$\Theta_x = \iint_D y^2 \rho(x, y) dD =$$

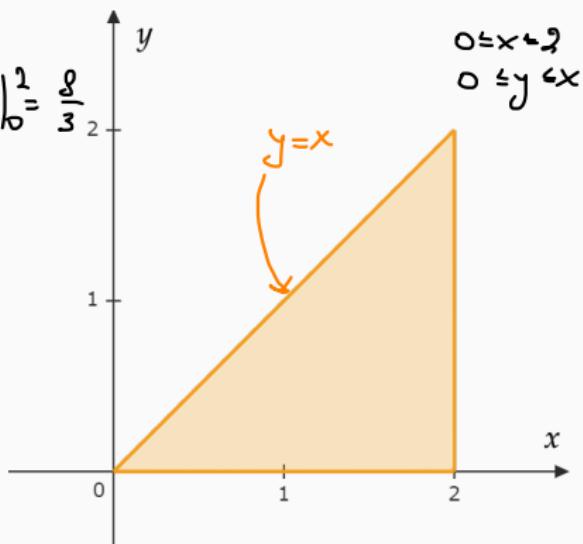
$$= \int_0^2 \int_0^{x^2} xy^3 dy dx = \int_0^2 \left(\frac{xy^4}{4} \right) \Big|_0^{x^2} dx = \int_0^2 \frac{x^5}{4} dx = \frac{x^6}{24} \Big|_0^2 = \frac{2}{3}$$

$$\Theta_y = \iint_D x^2 \rho(x, y) dD =$$

$$= \int_0^2 \int_0^{x^2} x^3 y dy dx = \frac{16}{3}$$

$$\Theta_0 = \Theta_x + \Theta_y =$$

? moment of inertia of an object
about origin - polar moment of
inertia



Computing moment of inertia

Exercise 12

Let K be a solid region bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$. Find the moments of inertia of the K about the yz -plane, the xz -plane, and the xy -plane.

Thank you!