Analysis III: Auditorium exercise class

Double integrals, Theorem of transfrmation, Center of mass and moment of inertia, Steiner's Theorem

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Fubini theorem:

- If f be integrable over D and for all $x \in [a_1, b_1], y \in [a_2, b_2]$ there exist integrals

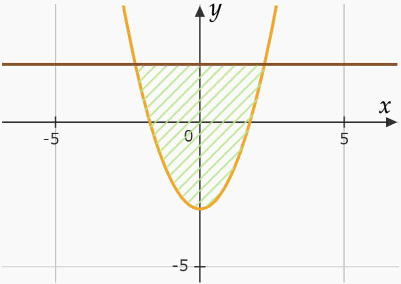
$$F(x) = \int_{a_2}^{b_2} f(x,y)\,dy \quad G(y) = \int_{a_1}^{b_1} f(x,y)\,dx$$

then it holds

$$\int_D f(x))dx = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)\,dx\,dy$$

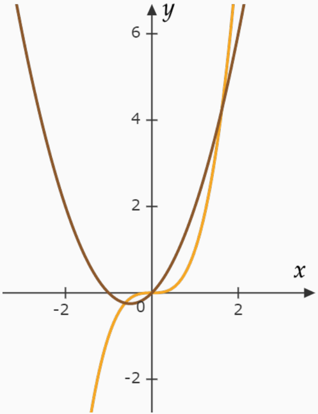
Exercise 1

Determine the area of the region between the two curves $y = x^2 - 2$ and $y = 2$.



Exercise 2

Determine the area of the region between the two curves $y = x^3$ and $y = x^2 + x$.



Exercise 3

Compute the volume of the body D defined by $0 < x < 1$, $-x < y < x$ and $0 < z < 3 + x^2 - 2y$.

Exercise 4

Compute the volume of space region D bounded by the surfaces $x^2 + y^2 = 1$, $z = 0$ and $z = -y$.

Coordinate Transformations

Theorem of transformation

Let $U \subset \mathbb{R}^n$ be compact, measurable set, $\Phi : D \to \mathbb{R}^n$ be C^1-coordinate transform. So the transformation Φ is invertible on D^0. Let further $K = \Phi(D)$. Then for continuous function $f : K \subset \mathbb{R}^n \to \mathbb{R}$ it holds

$$\int_K f(\mathbf{x})\, d\mathbf{x} = \int_D f(\Phi(\mathbf{u})) \cdot |\det\,(J\Phi(\mathbf{u})|\, d\mathbf{u}$$

Exercise 5

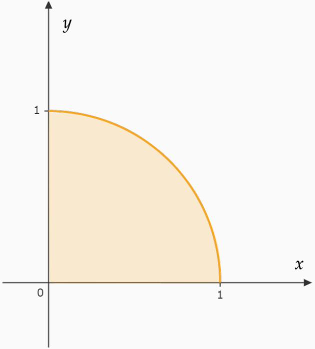
Compute the following integral

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} (x^2 + y^2)\, dydx$$

Solution:

$x = r\cos\phi$

$y = r\sin\phi$



Exercise 6

Calculate the integral

$$\iiint_U e^{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\, dxdydz,$$

where the region U is the unit ball $x^2 + y^2 + z^2 \leq 1$.

Exercise 7

Calculate the integral

$$\iiint\limits_U \left(x^2+y^2\right) dxdydz,$$

where the region U is bounded by the surfaces $x^2+y^2=3z, z=3$.

Application

Let $K \subset \mathbb{R}^3$ be the body with the continuous *mass density function* $\rho : K \to \mathbb{R}$.

- The mass M of the body K is given by

$$M = \int_K \rho(x, y, z)\, dK$$

- The center of mass of K is computed as

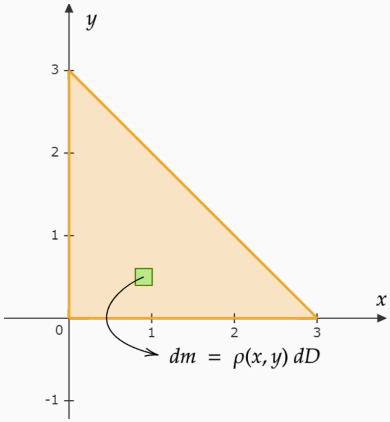
$$\mathbf{x_s} = \begin{pmatrix} x_s \\ y_s \\ z_s \end{pmatrix} = \frac{1}{M} \begin{pmatrix} \int_K \rho(x, y, z) \cdot x\, dK \\ \int_K \rho(x, y, z) \cdot y\, dK \\ \int_K \rho(x, y, z) \cdot z\, dK \end{pmatrix}$$

Computing the total mass

Exercise 8

Consider a triangular domain D with vertices $(0,0), (0,3), (3,0)$ and with density $\rho(x,y) = xy$. Find the total mass of D.

$$M = \iint_D dm = \iint_D \rho(x,y) dD$$

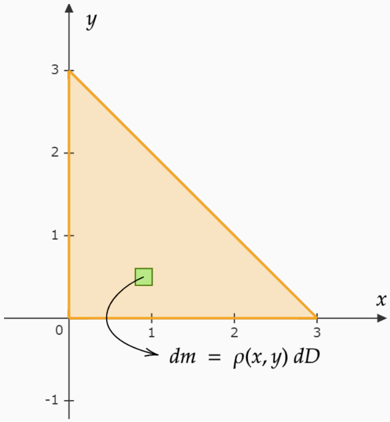


Exercise 9

Consider the same triangular domain D with vertices $(0,0)$, $(0,3)$,$(3,0)$ and with density $\rho(x,y) = xy$. Find the center of mass.

$$x_S = \frac{\iint\limits_R x\rho(x,y)\,dD}{\iint\limits_R \rho(x,y)\,dD} =$$

$$y_S = \frac{\iint\limits_R y\rho(x,y)\,dD}{\iint\limits_R \rho(x,y)\,dD} =$$



Exercise 10

Sketch the quarter sphere K given by $y \leq 0, z \leq 0, x^2 + y^2 + z^2 \leq 16$ and compute its center of mass for the given density function $\rho = x^2 + y^2 + z^2 + 1$ using polar coordinates.

Application

- The moment of inertia of K with respect to the given *axis of rotation* A is given by

$$\Theta_A = \int\limits_K \rho(x,y,z) r^2(x,y,z)\, dK,$$

where $r(x,y,z)$ is the distance from the point $(x,y,z)^T \in K$ to the axis A.

Theorem (Steiner's Theorem)

Let S be the axis parallel to A going through the center of the mass $\mathbf{x_s}$ of a body K, d the distance to the axis A from the point $\mathbf{x_s}$, M - mass of K, ρ - constant density of K. Then the moment of inertia w.r.t A is given by

$$\Theta_A = Md^2 + \Theta_S$$

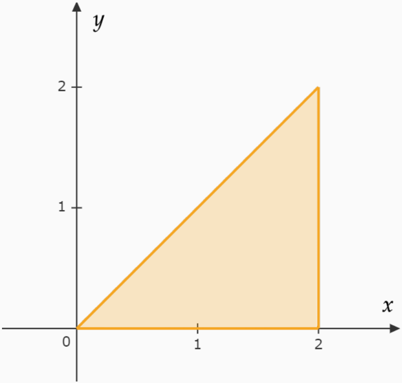
Exercise 11

Consider the triangular region D with vertices $(0,0)$, $(2,0)$,$(2,2)$ and with density $\rho(x,y) = xy$. Find the moments of inertia w.r.t axes.

$\Theta_x = \iint_D y^2 \rho(x,y)\, dD =$

$\Theta_y = \iint_D x^2 \rho(x,y)\, dD =$

$\Theta_0 = \Theta_x + \Theta_y =$



Exercise 12

Let K be a solid region bounded by $x + 2y + 3z = 6$ and the coordinate planes with density $\rho(x, y, z) = x^2yz$. Find the moments of inertia of the K about the yz-plane, the xz-plane, and the xy-plane.

Thank you!