Analysis III: Auditorium exercise class

Implicit representation of curves and surfaces, singular points, Constrained minimization problems, Lagrangian Double integrals

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December 6, 2021

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- RECOVERED
- TESTED

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If you cannot prove this, please leave the room now.
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Last Class: Implicitly Defined Functions

Consider a system of nonlinear equations

implicit function

$$g(x) = 0,$$

with $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$, $m < n$, i.e more unknowns than equations. - underdetermined system of equations.

We want to solve such systems locally expressing some variables via other.

Last Class: Implicit Function Theorem

Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 - function. Let $(x, y) \in D$, where $x \in \mathbb{R}^{n-m}, y \in \mathbb{R}^m$. Let $(x_0, y_0) \in D$ - solution to $g(x_0, y_0) = 0$. If the Jacobian matrix

$$\frac{\partial g}{\partial y}(x_0, y_0) := \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \dots & \frac{\partial g_1}{\partial y_m} \\ \dots & & \dots \\ \frac{\partial g_m}{\partial y_1} & \dots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}$$

is regular, then there exist neighbourhoods U of x_0, V of $y_0, U \times V \subset D$ and a uniquely determined continuous differentiable function $f : U \to V : f(x_0) = y_0$ and $g(x, f(x)) = 0$ for all $x \in U$ and

$$J\,f(x) = -\left(\frac{\partial g}{\partial y}(x, f(x))\right)^{-1} \left(\frac{\partial g}{\partial x}(x, f(x))\right)$$

Representation of curves

- Explicit : $y = g(x)$
- Implicit: $g(x,y) = 0$

locally!

– Implicit function theorem $\Longrightarrow$ if

Here (x,y) - a fixed point in which we want to find an explicit form

$$\text{grad } g(x,y) = (g_x, g_y) \neq 0$$

then $g(x,y)$ locally defines function

resp. if $g_x \neq 0$

if $g_y \neq 0$ → $y = f(x)$ or $x = \bar{f}(y)$

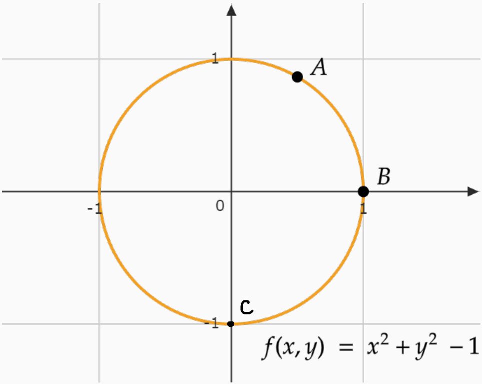
- Let $g(x_0, y_0) = 0$. Then if

$$\text{grad } (x_0, y_0) = 0,$$

→ $g_x = 0$ and $g_y = 0$

(x_0, y_0) is called singular point.

Example: Implicit representation of a circle (locally!)



One cannot represent the circle as a function of one variable, because for each choice of $x \in (-1;1)$ there are two choices of y:
$\pm\sqrt{1-x^2}$ => not a function.

But
Implicit F. theorem:
$\exists$ local representation

- B - no represent. as $y(x)$, since
  $f_y = 2y|_{(1,0)} = 0$
  but, $\exists$ as $x(y)$, since
  $f_x = 2x|_{(1,0)} = 2 \neq 0$
  + vert. tangent

- C: $f_x = 2x|_{(0,-1)} = 0$
  $f_y = 2y|_{(0,-1)} = -2 \neq 0$
  => $\exists$ repr. as $y(x)$ at C.
  + hor. tangent

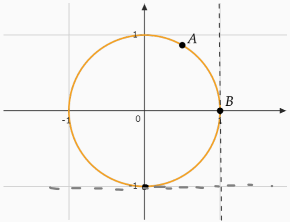
Regular points

- The point (x_0, y_0) is called regular point if

$$\operatorname{grad} g(x_0, y_0) \neq 0.$$

+ belongs to the solution set: $g(x_0,y_0)=0$!

- At regular points the set of solutions is described by a contour line:



- $g_x(x_0, y_0) = 0,\ g_y(x_0, y_0) \neq 0$
  - horizontal tangent at (x_0, y_0)
- $g_x(x_0, y_0) \neq 0,\ g_y(x_0, y_0) = 0$
  - vertical tangent at (x_0, y_0)

Classification of singular points

grad $g(x_0, y_0) = 0$

A singular point (x_0, y_0) is called

- isolated point if

$$\det Hg(x_0, y_0) > 0$$

- double point if

$$\det Hg(x_0, y_0) < 0$$

- return point (cusp) if

$$\det Hg(x_0, y_0) = 0$$

Note that point (x_0, y_0) should belong to the solution set of g, i.e $g(x_0, y_0) = 0$

Example: Singular points

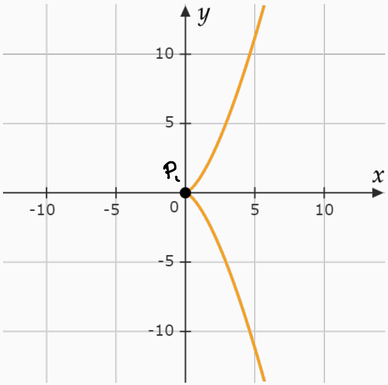


Figure 1: $f(x, y) = x^3 - y^2$,
$f(x, y) = 0$

Compute all singular points of f and determine their type.

$\operatorname{grad} f(x,y) = (3x^2, -2y)$

$\begin{cases} 3x^2 = 0 \\ -2y = 0 \end{cases} \Rightarrow P_1 = (0,0)$ - candidate for a singular pt.

Need also to check if P_1 satisfies $f(P_1) = 0$,

$f(P_1) = 0 - 0 = 0$ ✓

Hence P_1 – singular point.

$Hf(x,y) = \begin{pmatrix} 6x & 0 \\ 0 & -2 \end{pmatrix}$

$\det(Hf(P_1)) = 0$

$\Rightarrow P_1$ is a cusp.

Example: Singular points 2

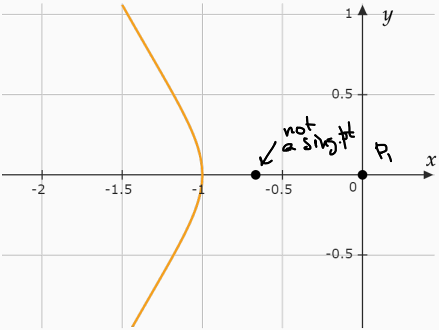


Figure 2:
$f(x, y) = x^3 + x^2 + y^2, f(x, y) = 0$

Singular points:

$\text{grad}\, f(x,y) = (3x^2 + 2x,\ 2y)$

$\begin{cases} 3x^2 + 2x = 0 \\ 2y = 0 \end{cases} \Leftrightarrow \begin{cases} x(3x+2) = 0 \\ y = 0 \end{cases}$

$\left[\begin{array}{l} \begin{cases} y = 0 \\ x = 0 \end{cases} \quad P_1(0,0) \\ \begin{cases} y = 0 \\ x = -\frac{2}{3} \end{cases} \quad P_2\left(-\frac{2}{3}, 0\right) \end{array}\right.$

P_1, P_2 - candidates for sing. pts

$f(P_1) = 0 + 0 + 0 = 0$ ✓

$f(P_2) = \left(-\frac{2}{3}\right)^3 + \left(-\frac{2}{3}\right)^2 + 0 = -\frac{8}{27} + \frac{4}{9} \neq 0$

$\Rightarrow P_2$ is not a sing. pt.

Check the type of P_1:

$Hf(x,y) = \begin{pmatrix} 6x+2 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(Hf(P_1)) = 4 > 0 \Rightarrow$ isolated point

Example: Singular points 3

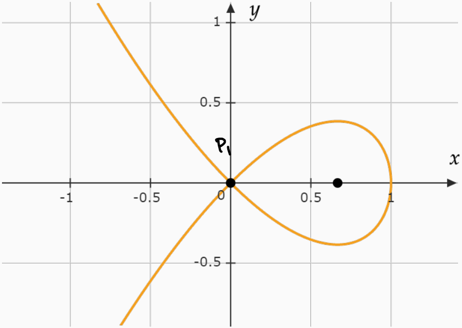


Figure 3:
$f(x, y) = x^3 - x^2 + y^2, f(x, y) = 0$

Singular pts:

grad $f(x,y) = (3x^2 - 2x, 2y)$

$\begin{cases} x(3x-2) = 0 \\ 2y = 0 \end{cases}$ $\left[\begin{array}{l} \begin{cases} x = 0 \\ y = 0 \end{cases} \\ \begin{cases} 3x - 2 = 0 \\ y = 0 \end{cases} \end{array}\right.$

$P_1 = (0,0)$ $P_2 = (2/3, 0)$

$f(0,0) = 0^3 - 0^2 + 0^2 = 0$ ✓

$f(P_2) \neq 0$ ✗

$Hf(x,y) = \begin{pmatrix} 6x-2 & 0 \\ 0 & 2 \end{pmatrix}$

$\det(Hf(P_1)) = -4 < 0$

$\Rightarrow$ double point

Exercise 1

Consider the curve

$$f(x,y) = x^2 + y^2 - 4 = 0.$$

Find

- symmetries
- tangent lines (horizontal and vertical) for regular points
- singular points and determine their type

Symmetry: • $f(x,y) = f(-x,y)$ — w.r.t. y-axis
• $f(x,y) = f(x,-y)$ — w.r.t. x-axis
• $f(x,y) = f(-x,-y)$ — w.r.t. origin

\+ $f(x,y) = f(y,x)$

1. $f(x,y) = x^2+y^2-4=0$

$f(x,-y) = x^2+y^2-4$

$f(x,y)=f(x,-y)=f(-x,y)=f(-x,-y)$ - since there is power of 2

$\Rightarrow$ f is symmetric w.r.t x-axis, y-axis and $(0,0)$.

2. Tangent lines

– Horizontal tangent: $f_x(x_0,y_0)=0$ $\quad f_y(x_0,y_0)\neq 0$ $\quad f(x_0,y_0)=0$

$$\begin{cases} f_x = 2x = 0 \\ f_y = 2y \neq 0 \\ f(x,y) = x^2+y^2-4=0 \end{cases} \qquad \begin{cases} x=0 \\ y\neq 0 \\ f(0,y) = 0+y^2-4=0 \end{cases} \Rightarrow y=\pm 2 \neq 0$$

$P_1=(0,2)$; $P_2=(0,-2)$.

$\Rightarrow$ There exist horizontal tangents at points $P_1=(0,2)$; $P_2=(0,-2)$.

– Vertical tangent

$$\begin{cases} f_x = 2x \neq 0 \\ f_y = 2y = 0 \\ f(x,y)=0 \end{cases} \qquad \begin{cases} x\neq 0 \\ y=0 \\ f(x,0) = x^2-4=0 \end{cases} \qquad \begin{cases} x=\pm 2 \\ y=0 \\ x\neq 0 \end{cases}$$

$\Rightarrow \exists$ vertical tangents at pts $P_3(0,2)$, $P_4(0,-2)$.

• Singular points: $\operatorname{grad} f(x,y) = (2x,2y)$ $\Rightarrow$ $P(0,0)$

$f(0,0) = 0+0-4$ $\Rightarrow$ P is <u>not</u> a sing. pt

$\Rightarrow$ there is no singular point for f.

// we knew it even before, since $f(x,y)$ is a circle $R=2$, $C(0,0)$ $\Rightarrow$ no sing. pts.

Exercise 2

Consider the curve

$$f(x, y) = x^3 + y^3 - xy$$

Find

- symmetries
- tangent lines (horizontal and vertical) for regular points
- singular points and determine their type

$f(x,y) = x^3 + y^3 - xy$

$\operatorname{grad} f(x,y) = (3x^2 - y,\ 3y^2 - x)$

- Singular pts:

$$\begin{cases} 3x^2 - y = 0 \\ 3y^2 - x = 0 \end{cases} \qquad \begin{cases} y = 3x^2 \\ 3(3x^2)^2 - x = 0 \end{cases} \qquad \begin{cases} 27x^4 - x = 0 \\ y = 3x^2 \end{cases} \qquad \begin{cases} x = 0 \text{ or } x = 1/3 \\ y = 3x^2 \end{cases}$$

$\Rightarrow P_1 = (0,0)\ ;\ P_2 = (1/3, 1/3)$

Check if P_1, P_2 belong to the solution set:

$f(P_1) = 0 + 0 - 0 \cdot 0 = 0$ ✓

$\Rightarrow P_1$ is a singular pt

$f(P_2) = 1/3^3 + \frac{1}{3^3} - \frac{1}{3} \cdot \frac{1}{3}$ ✗

P_2 is <u>not</u> a sing. pt.

$Hf(x,y) = \begin{pmatrix} 6x & -1 \\ -1 & 6y \end{pmatrix}$ $\qquad \det(Hf(0,0)) = \det\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = -1 < 0 \Rightarrow$ double point.

- Tangent lines

Horizontal:
$$\begin{cases} f_x = 0 = 3x^2 - y \\ f_y = 3y^2 - x \neq 0 \\ f(x,y) = 0 \end{cases} \qquad \begin{cases} y = 3x^2 \\ f(x, 3x^2) = x^3 + (3x^2)^3 - 3x^3 = 27x^6 - 2x^3 = 0 \\ 3y^2 - x \neq 0 \end{cases}$$

$\Downarrow$

$x^3(27x^2 - 2) = 0$

$x = 0$ or $x = \sqrt{\frac{2}{27}}$

$\Rightarrow P_1 = (0,0) \qquad P_2 = \frac{1}{3}\begin{pmatrix} \sqrt[3]{9} \\ 2^{2/3} \end{pmatrix}$

$f_y(P_1) = 0 \Rightarrow P_1$ is a singular pt

$f_y(P_2) \neq 0 \Rightarrow P_2$ is a point with a horiz. tangent

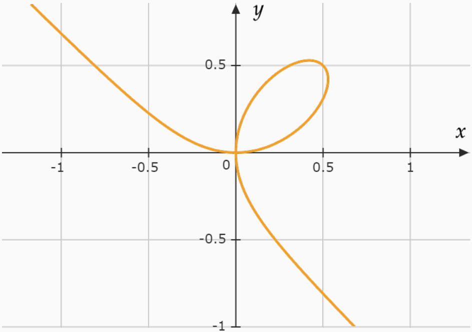


Figure 4: $f(x, y) = x^3 - x^3 + xy = 0$

- vertical

$$\begin{cases} f_y = 3y^2 - x = 0 \\ f_x = 3x^2 - y \neq 0 \\ f(x,y) = x^3 + y^3 - xy \end{cases}$$

$$\begin{cases} x = 3y^2 \\ f_x \neq 0 \\ f(3y^2, y) = 27y^6 + y^3 - 3y^3 \underset{=}{=} 0 \end{cases}$$

$$\begin{cases} y = 0 \text{ or } y = \frac{\sqrt[3]{2}}{3} \\ x = 3y^2 \end{cases}$$

$P_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ – singular

$P_3 = \frac{1}{3}\begin{pmatrix} 2^{2/3} \\ \sqrt[3]{2} \end{pmatrix}$ – vertical tangent

Unconstrained minimization problem

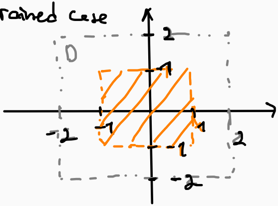
or $f: D \supset \mathbb{R}^n \to \mathbb{R}$, $\min\limits_{x \in D} f$. D - domain of f — unconstrained case

e.g.: $D: [-2,2] \times [-2,2]$

But if $\min\limits_{G} f$ => constrained

→ $G: [-1,1] \times [-1,1]$

G - admissible set



Determine the minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$

Notation:

$$\min_{x \in \mathbb{R}^n} f(x), \tag{1}$$

- for maximum: $-f(x)$ in (1)

Determine the minimum of the function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ under the constraint

$$\begin{pmatrix} g_1(x) \\ \vdots \\ g_m(x) \end{pmatrix} = g(x) = 0,$$

equality constraint

$h(x) \leq 0$ => do not consider in this class!

where $g : D \to \mathbb{R}^m$.

Notation:

$$\min_{x \in G} f(x),$$

where

$$G := \{x \in D : g(x) = 0\} \subset D$$

admissible set – points that satisfy the constraints

Example of a constrained problem

or $f(x,y) = \min!$

Let $f : \mathbb{R}^2 \to \mathbb{R}$, given the problem

$$f(x, y) = x^4 - 2xy + 3 \to \min$$

subject to (s.t.)

$$g(x, y) = x - y = 0$$

constraint

The Lagrangian

Lagrange f. is a function we use to generalize the optimality conditions we had to the constrained case.

- The Lagrange–function is defined as

Sometimes defined as $F(x, \lambda)$, then $\nabla F = \begin{pmatrix} F_x \\ F_y \\ \vdots \\ F_\lambda \end{pmatrix}$

$$F(x) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \tag{2}$$

where $\lambda = (\lambda_1, ..., \lambda_m)^T$ - Lagrange multipliers.

- A necessary condition for existence of local extrema:

$$\begin{pmatrix} F_{x_1} \\ \vdots \\ F_{x_m} \end{pmatrix} = \text{grad } F(x) = 0 \tag{3}$$

$+ \; g(x) = 0$

The Lagrange Multiplier Rule

Let $x_0 = (x_{1_0}, ...x_{n_0}) \in D$ – local extremum of f that satisfies constraint: $g(x_0) = 0$.

- If the following regularity condition is satisfied

in German rang

$$\text{rank } (Jg(x_0)) = m,$$

matrix has a full rank
If the vectors are linearly independent

(i.e the Jacobian matrix has a full rank)
then there exist Lagrange multipliers $\lambda_1, ..\lambda_m$ of a Lagrangian (2) such that the necessary optimality condition (3) is satisfied:

$$\text{grad } F(x_0) = 0$$

=> x_0 - candidate for optimum

Sufficient optimality conditions (of 2nd order)

↳ conditions for x_0 to be an optimum.

- If rank $(Jg(x_0)) = m$ for $x_0 \in G$ and grad $F(x_0) = 0$ and $HF(x_0)$ is **positive definite** (negative) on tangential space

$$TG(x_0) := \{w \in \mathbb{R}^n : \langle \text{grad } g_i, w \rangle = 0\},$$

i.e.

$$w^T HF(x_0) w > 0 \text{ for } w \in TG(x_0 \setminus 0)$$

then x_0 is a **strict local minimum** (maximum) of f that satisfies the constraints g.

Remark

- Find stationary points of the Lagrangian,
- evaluate f in them $\Rightarrow$ one with the largest - max
  smallest - min.

- If an admissible set G is compact and function f is a continuous, then f attains its max/min on G:
  - the stationary point where function has the largest value - global maximum
  - the stationary point where function has the smallest value - global minimum

$\implies$ no need to check the second order optimality condition.

Exercise

Find the global extrema of the function

$$f(x, y, z) = x - 8y + z.$$

on the admissible set defined by $g(x, y, z) = 0$ and $h(x, y, z) = 0$:

sphere $R=5, C(0,-4,0)$ $g(x, y, z) = x^2 + (y+4)^2 + z^2 - 25$

sphere $R=3$ $C(0,0,0)$ $h(x, y, z) = x^2 + y^2 + z^2 - 9$

$\Downarrow$ the admissible set is closed and bounded in $\mathbb{R}^3$

$\Rightarrow$ is a compact set

$\Rightarrow$ can use remark from slide 20.

- checking the regularity condition

$$J(g,h)(x,y,z) = \begin{pmatrix} g_x & g_y & g_z \\ h_x & h_y & h_z \end{pmatrix} = \begin{pmatrix} 2x & 2(y+4) & 2z \\ 2x & 2y & 2z \end{pmatrix}$$

Check if $J(g,h)$ has a full rank.

$$\alpha_1 \begin{pmatrix} 2x \\ 2(y+4) \\ 2z \end{pmatrix} + \alpha_2 \begin{pmatrix} 2x \\ 2y \\ 2z \end{pmatrix} = 0$$

Vectors x, y - linearly independent iff $(\alpha_1 x + \alpha_2 y = 0 \Leftrightarrow \alpha_1 = 0 \wedge \alpha_2 = 0)$

$\Downarrow$ full rank

$$\begin{cases} \alpha_1 = -\alpha_2 \vee x = 0 \\ \alpha_1 = \dfrac{\alpha_2 y}{y+4} \\ \alpha_1 = -\alpha_2 \vee z = 0 \end{cases}$$

$\alpha_1 = -\alpha_2$ into 2 $\Rightarrow -(y+4)\alpha_2 + 2y\alpha_2 = 0 \Rightarrow \alpha_2 = 0$ or $y = 0$

$\Downarrow$

$\alpha_1 = 0$

$$\left.\begin{array}{l} x = 0 \Rightarrow \alpha_1 = -\alpha_2 \neq 0 \\ z = 0 \Rightarrow \alpha_1 = -\alpha_2 \neq 0 \end{array}\right\} \text{bad case}$$

$\Rightarrow$ matrix has a full rank on $\mathbb{R}^3 \setminus (0,y,0)$.

But lets check if points $(0,y,0)$ belong to the solution set:

$0 = g(0,y,0) = 0 + (y+4)^2 + 0 - 25 \Rightarrow y = 1, -9$

$0 = h(0,y,0) = 0 + y^2 + 0 - 9 \Rightarrow y = \pm 3$

$\Rightarrow$ not the same $\Rightarrow$ does not belong to $(S) \Rightarrow J\binom{h}{g}$ has full rank on the whole solution set.

$\Rightarrow$ Regularity condition holds in all admissible points.

Lagrangian:

$$F(x) = f(x) + \lambda_1 g(x) + \lambda_2 h(x)$$

$$F(x,y,z) = x - 8y + z + \lambda_1 \left(x^2 + (y+4)^2 + z^2 - 25 \right) + \lambda_2 \left(x^2 + y^2 + z^2 - 9 \right)$$

To find a stationary pt compute derivatives $\overset{\circ}{=}$ + $g(x)=0$; $h(x)=0$:

$$\begin{cases} F_x(x,y,z) = 1 + 2\lambda_1 x + 2\lambda_2 x = 0 \\ F_y(x,y,z) = -8 + 2(y+4)\lambda_1 + 2\lambda_2 y = 0 \\ F_z(x,y,z) = 1 + 2\lambda_1 z + 2\lambda_2 z = 0 \\ g(x) = 0 \\ h(x) = 0 \end{cases}$$

system of equations solving which we obtain stationary pts

$$\begin{cases} 1 \quad 2x(\lambda_1 + \lambda_2) = -1 \\ 2 \quad y(2\lambda_1 + 2\lambda_2) + 8\lambda_1 - 8 = 0 \\ 3 \quad 2z(\lambda_1 + \lambda_2) = -1 \\ 4 \quad x^2 + (y+4)^2 + z^2 - 25 = 0 \\ 5 \quad x^2 + y^2 + z^2 - 9 = 0 \end{cases}$$

"4" − "5": $8y + 16 - 25 + 9 = 0$

$y = 0$

plug into "2":

$8\lambda_1 = 8 \Rightarrow \lambda_1 = 1$

"1" = "3": $x(\lambda_1 + \lambda_2) = z(\lambda_1 + \lambda_2)$

$(x - z)(\lambda_1 + \lambda_2) = 0$

$x = z$ or $\lambda_1 = -\lambda_2$

$$\begin{cases} y = 0 \\ \lambda_1 = 1 \\ x = z \quad \text{or} \quad \lambda_1 = -\lambda_2 \\ x^2 + y^2 + z^2 - 9 = 0 \\ 2z(\lambda_1 + \lambda_2) = -1 \end{cases}$$

$$\begin{cases} x = z \\ y = 0 \\ \lambda_1 = 1 \\ x^2 + y^2 + z^2 - 9 = 0 \\ 2z(\lambda_1 + \lambda_2) = -1 \end{cases} \Rightarrow \begin{cases} z = \pm\sqrt{9/2} \\ y = 0 \\ \lambda_1 = 0 \\ 2x^2 = 9 \Rightarrow x = \pm\sqrt{9/2} \\ \lambda_2 = \left(\mp \frac{1}{2}\sqrt{\frac{2}{9}}\right) \end{cases}$$

$$\begin{cases} \lambda_1 = -\lambda_2 \\ y = 0 \\ \lambda_1 = 1 \\ x^2 + y^2 + z^2 - 9 = 0 \\ 2z(\lambda_1 + \lambda_2) = -1 \end{cases} \Rightarrow 0 = -1 \quad \lightning$$

So we have points

$$P_1 = \begin{pmatrix} \frac{3}{\sqrt{2}} \\ 0 \\ \frac{3}{\sqrt{2}} \end{pmatrix} \qquad P_2 = \begin{pmatrix} -\frac{3}{\sqrt{2}} \\ 0 \\ -\frac{3}{\sqrt{2}} \end{pmatrix}$$

with $\lambda_1 = 0$, $\lambda_2 = -\frac{\sqrt{2}}{6}$ with $\lambda_1 = 0$, $\lambda_2 = +\frac{\sqrt{2}}{6}$

$f(P_1) = 3\sqrt{2}$ $f(P_2) = -3\sqrt{2}$

$\Rightarrow P_1$ – max of f ; P_2 – minimum of f.

Can do this only because admissible set is a compact

Exercise

Given the function $f : \mathbb{R}^3 \to R$

$$f(x, y, z) = z.$$

Compute all extrema of the function that satisfy the following constraints

$$g_1(x, y, z) = x^2 + y^2 - 9$$
$$g_1(x, y, z) = y - z$$

– cylinder => closed, bounded => compact

– plane

and determine whether it is maximum/minimum using the Lagrange multiplier rule.

1. Check the regularity condition

$$J\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}(x,y,z) = \begin{pmatrix} 2x & 2y & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\alpha_1 \begin{pmatrix} 2x \\ 2y \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = 0 \Rightarrow \begin{cases} \alpha_1 = 0 \ \vee \ x = 0 \\ \alpha_2 = -2\alpha_1 y \Rightarrow \alpha_1 = 0 \vee y = 0 \\ \alpha_2 = 0 \end{cases} \Rightarrow \boxed{x \neq 0 \wedge y \neq 0}$$

$\Rightarrow$ the matrix has a full rank on the set $\mathbb{R}^3 \setminus \langle 0,0,z \rangle$, z arbitrary.

$\langle 0,0,z \rangle \notin$ admissible set $\Rightarrow J\begin{pmatrix} g_1 \\ g_2 \end{pmatrix}$ has a full rank on the adm. set.

$\Rightarrow$ Reg. cond is satisfied $\Rightarrow \exists \lambda_1, \lambda_2$ s.t grad $F(x_0) = 0$

2. Lagrangian

$$F(x,y,z) = z^2 + \lambda_1 (x^2 + y^2 - 9) + \lambda_2 (y - z)$$

3. Lagrange Multiplier Rule

$$\begin{pmatrix} F_x \\ F_y \\ F_z \\ g_1 \\ g_2 \end{pmatrix} = \begin{pmatrix} 2\lambda_1 x \\ 2\lambda_1 y + \lambda_2 \\ 2z - \lambda_2 \\ x^2+y^2-9 \\ y-z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{cases} x=0 \text{ or } \lambda_1 = 0 \\ 2\lambda_1 y + \lambda_2 = 0 \\ \lambda_2 = 2z \\ x^2+y^2 = 9 \\ y = z \end{cases}$$

$$\begin{cases} x=0 \\ 2\lambda_1 y + \lambda_2 = 0 \\ \lambda_2 = 2z \\ x^2+y^2=9 \\ y=z \end{cases} \Leftrightarrow \begin{cases} x=0 \\ \lambda_1 = -\lambda_2/2y = \mp 3/\pm 3 = -1 \\ \lambda_2 = \pm 6 \\ 0+y^2=9 \Rightarrow y = \pm 3 \\ z = \pm 3 \end{cases} \Rightarrow P_1 = \begin{pmatrix} 0 \\ 3 \\ 3 \end{pmatrix} \text{ with } \lambda_1=-1,\ \lambda_2 = 6 \qquad P_2 = \begin{pmatrix} 0 \\ -3 \\ -3 \end{pmatrix} \ \lambda_1 = -1,\ \lambda_2 = -6$$

$$\begin{cases} \lambda_1=0 \\ 2\lambda_1 y + \lambda_2 = 0 \\ \lambda_2 = 2z \\ x^2+y^2=9 \\ y=z \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 0 \\ \lambda_2 = 0 \\ z = 0 \\ x = \pm 3 \\ y = 0 \end{cases} \Rightarrow P_3 = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} \ \lambda_1 = \lambda_2 = 0 \qquad P_4 = \begin{pmatrix} -3 \\ 0 \\ 0 \end{pmatrix} \ \lambda_1 = \lambda_2 = 0$$

Note that for P_3, P_4 $\lambda_1 = \lambda_2 = 0$ i.e the constraint is inactive.

4. Tangential space

$TG(x_0) = \{ \text{grad}\,(g)(x_0) \cdot w = 0 \}$

P_1: $\begin{pmatrix} 6 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$ $\quad \begin{cases} 6w_1 = 0 \\ w_2 = w_3 \end{cases} \Rightarrow \alpha(0,1,1)$

$\Rightarrow TG(P_1) = \{ w \in \mathbb{R}^n : \alpha(0,1,1) \}$

— tangent space at point P_1

$x^T A x \geq 0 \quad \forall x$

$\Rightarrow A$ - pos.def / nes

// similarly for other points

5. Checking sufficient cond:

$HF(x,y,z) = \begin{pmatrix} 2\lambda_1 & 0 & 0 \\ 0 & 2\lambda_2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$

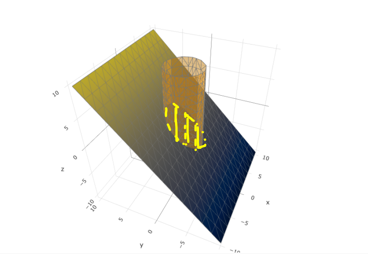
Plugging in λ_1, λ_2, w for P_1:

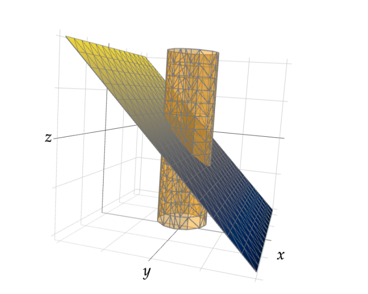
$\alpha^2 (0,1,1) \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \alpha^2 (0,0,2) \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = 2\alpha^2 > 0 \quad \forall \alpha$

$\Rightarrow HF(P_1)$ is pos. def on the $TG(P_1) \Rightarrow P_1$ - strict local minimum.

Analogously for other pts.

Exercise

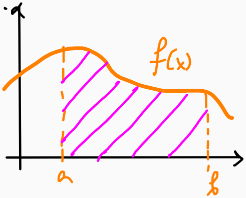




Double Integrals

Area/volume integrals - calculate the volume/area under the graph of $f(x,y)$

$$V = \int_D f(x)dx$$



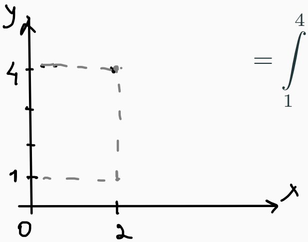
Example 1

Note: used Fubini here!

Integrate the function $f(x,y) = xy$ over the rectangle $[0,2] \times [1,4]$.

$$\int_D f(x,y)\,dx = \int_1^4 \int_0^2 x \cdot y\,dx\,dy = \int_1^4 \left(y \cdot \frac{x^2}{2}\right)\Big|_0^2 dy$$

$$= \int_1^4 y\left(\frac{2^2}{2} - \frac{0^2}{2}\right)\,dy = \int_1^4 2y dy = 4^2 - 1^2 = 15$$



Start calculations by determining the ranges of x, y:

$0 \leq x \leq 2$
$1 \leq y \leq 4$

Fubini theorem:

- If f is integrable over D and for all $x \in [a_1, b_1], y \in [a_2, b_2]$ there exist integrals

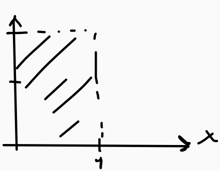
$$F(x) = \int_{a_2}^{b_2} f(x,y)\,dy \quad G(y) = \int_{a_1}^{b_1} f(x,y)\,dx$$

then it holds

$$\int_D f(x))dx = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)\,dx\,dy$$

Example 2

Projection on xy-plane



Integrate the function $f(x, y) = 2 - xy$ over the rectangle $[0, 1] \times [0, 2]$.

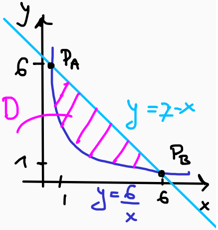
$0 \le x \le 1$
$0 \le y \le 2$

integrate w.r.t x

$$\int_D f(x, y)\, dx =$$

$$= \int_0^2 \int_0^1 2 - xy \, dx\, dy = \int_0^2 \left(2x - \frac{x^2 y}{2}\right)\Big|_0^1 dy = \int_0^2 2 - \frac{y}{2} \, dy = \left(2y - \frac{y^2}{4}\right)\Big|_0^2$$

$$= 4 - \frac{4}{4} = 3.$$



Compute the following integrals:

a) $\iint_D (x+y)\,dx\,dy$ over the region D bounded by curves $xy=6$ and $x+y=7$

Find intersection of these curves:

$$\begin{cases} xy = 6 \\ x+y=7 \end{cases} = \begin{cases} y = 6/x \\ y = 7-x \end{cases} \Rightarrow P_A = (1,6),\ P_B = (6,1)$$

First determine the ranges of x, y : $1 \le x \le 6$

$\frac{6}{x} \le y \le 7-x$ ↙ important!

$$\iint_D (x+y)\,dx\,dy = \int_1^6 \int_{6/x}^{7-x} x+y \,dx\,dy = \int_1^6 \left(xy + \frac{y^2}{2} \right)\Bigg|_{y=\frac{6}{x}}^{y=7-x} dx = \int_1^6 x(7-x) + \frac{(7-x)^2}{2} - x\frac{6}{x} - \frac{36}{2x^2}\,dx$$

Can use any order we want due to Fubini.

The outer – one with numbers

$$= \int_1^6 \left(-\frac{x^2}{2} - \frac{18}{x^2} + \frac{27}{2} \right) dx = \left(-\frac{x^3}{6} + \frac{18}{x} + \frac{27x}{2} \right) \Big|_1^6 = \frac{125}{3}.$$

2. $\iint_D (x-y)\,dx\,dy$ on the region bounded by $x = y^2$ and $x = \frac{y^2}{2} + 1$

$\hookrightarrow$ exercise later

Some more exercises

Consider the curve

$$f(x, y) = x^4 - x^2 + y^2$$

Find

- symmetries
- tangent lines (horizontal and vertical) for regular points
- singular points and determine their type

Some more exercises

Compute the global extrema of a function

$$f(x, y, z) = xy + z^2$$

subject to constraints

$$g(x, y, z) = x^2 + y^2 - 8 = 0$$

$$h(x, y, z) = x - y + 2z - 2 = 0$$

Thank you!