Analysis III: Auditorium exercise class

Implicit representation of curves and surfaces,
singular points,
Constrained minimization problems, Lagrangian
Surface integrals

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BITTE BEACHTEN SIE DIE 3G-REGEL!
PLEASE OBEY THE 3G RULE!

Zutritt zur Lehrveranstaltung haben nur:

- VOLLSTÄNDIG GEIMPFTE
- GENESENE
- GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen können, müssen Sie bitte den Raum jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted to persons who are:

- FULLY VACCINATED
- RECOVERED
- TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this, please leave the room now.
Otherwise you could be banned from the room!

Thank you for your understanding.
Protect yourself and others!

Last Class: Implicitly Defined Functions

Consider a system of nonlinear equations

$$g(x) = 0,$$

with $g : D \subset \mathbb{R}^n \to \mathbb{R}^m, m < n$, i.e more unknowns than equations. - underdetermined system of equations.

We want to solve such systems locally expressing some variables via other.

Last Class: Implicit Function Theorem

Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 - function. Let $(x, y) \in D$, where $x \in \mathbb{R}^{n-m}, y \in \mathbb{R}^m$. Let $(x_0, y_0) \in D$ - solution to $g(x_0, y_0) = 0$. If the Jacobian matrix

$$\frac{\partial g}{\partial y}(x_0, y_0) := \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \dots & \frac{\partial g_1}{\partial y_m} \\ \dots & & \dots \\ \frac{\partial g_m}{\partial y_1} & \dots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}$$

is regular, then there exist neighbourhoods U of x_0, V of $y_0, U \times V \subset D$ and a uniquely determined continuous differentiable function $f : U \to V : f(x_0) = y_0$ and $g(x, f(x)) = 0$ for all $x \in U$ and

$$J\, f(x) = -\left(\frac{\partial g}{\partial y}(x, f(x))\right)^{-1} \left(\frac{\partial g}{\partial x}(x, f(x))\right)$$

Representation of curves

- Explicit : $y = g(x)$
- Implicit: $g(x, y) = 0$

– Implicit function theorem $\implies$ if

$$\text{grad } g(x, y) = (g_x, g_y) \neq 0$$

then $g(x, y)$ locally defines function

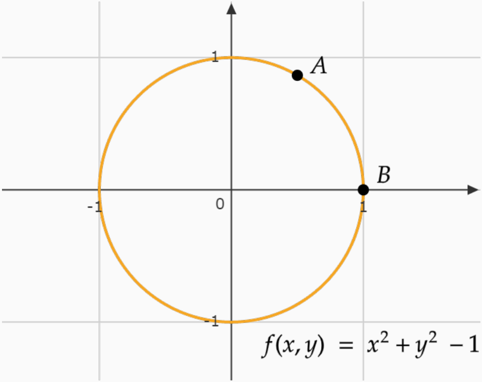
$$y = f(x) \text{ or } x = \bar{f}(y)$$

- Let $g(x_0, y_0) = 0$. Then if

$$\text{grad } (x_0, y_0) = 0,$$

(x_0, y_0) is called singular point.

Example: Implicit representation of a circle (locally!)

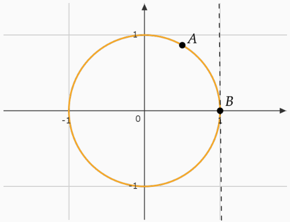


Regular points

- The point (x_0, y_0) is called regular point if

$$\text{grad}\,(x_0, y_0) \neq 0.$$

- At regular points the set of solutions is described by a contour line:



- $g_x(x_0, y_0) = 0,\ g_y(x_0, y_0) \neq 0$
  - horizontal tangent at (x_0, y_0)
- $g_x(x_0, y_0) \neq 0,\ g_y(x_0, y_0) = 0$
  - vertical tangent at (x_0, y_0)

Classification of singular points

A singular point (x_0, y_0) is called

- isolated point if

$$\det Hg(x_0, y_0) > 0$$

- double point if

$$\det Hg(x_0, y_0) < 0$$

- return point (cusp) if

$$\det Hg(x_0, y_0) = 0$$

Note that point (x_0, y_0) should belong to the solution set of g, i.e $g(x_0, y_0) = 0$

Example: Singular points

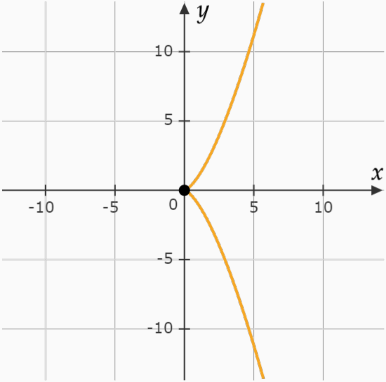


Figure 1: $f(x, y) = x^3 - y^2$,
$f(x, y) = 0$

Example: Singular points 2

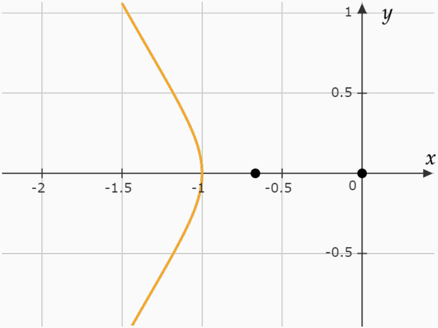


Figure 2:
$f(x, y) = x^3 + x^2 + y^2, f(x, y) = 0$

Example: Singular points 3

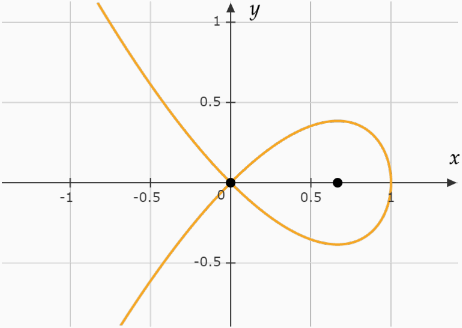


Figure 3:
$f(x, y) = x^3 - x^2 + y^2, f(x, y) = 0$

Exercise 1

Consider the curve

$$f(x, y) = x^2 + y^2 - 4 = 0.$$

Find

- symmetries
- tangent lines (horizontal and vertical) for regular points
- singular points and determine their type

Exercise 2

Consider the curve

$$f(x, y) = x^3 + y^3 - xy$$

Find

- symmetries
- tangent lines (horizontal and vertical) for regular points
- singular points and determine their type

Exercise 2 (cont'd)

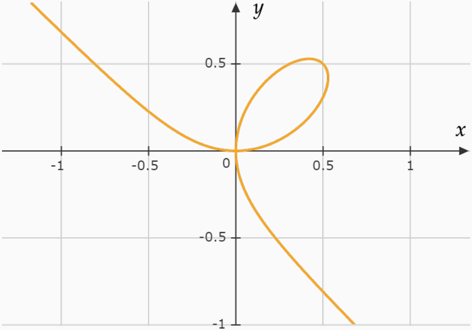


Figure 4: $f(x, y) = x^3 - x^3 + xy = 0$

Unconstrained minimization problem

Determine the minimum of the function $f : \mathbb{R}^n \to \mathbb{R}$

Notation:

$$\min_{x \in \mathbb{R}^n} f(x), \tag{1}$$

- for maximum: $-f(x)$ in (1)

Constrained minimization problem (equality constraints)

Determine the minimum of the function $f : D \subset \mathbb{R}^n \to \mathbb{R}$ under the constraint

$$g(x) = 0,$$

where $g : D \to \mathbb{R}^m$.

Notation:

$$\min_{x \in G} f(x),$$

where

$$G := \{x \in D : g(x) = 0\} \subset D$$

Example of a constrained problem

Let $f : \mathbb{R}^2 \to \mathbb{R}$, given the problem

$$f(x, y) = x^4 - 2xy + 3 \to \min$$

subject to

$$g(x, y) = x - y = 0$$

The Lagrangian

- The Lagrange–function is defined as

$$F(x) := f(x) + \sum_{i=1}^{m} \lambda_i g_i(x), \tag{2}$$

where $\lambda = (\lambda_1, ..., \lambda_m)^T$ - Lagrange multipliers.

- A necessary condition for existence of local extrema:

$$\text{grad } F(x) = 0 \tag{3}$$

The Lagrange Multiplier Rule

Let $x_0 = (x_{1_0}, \ldots x_{n_0}) \in D$ – local extremum of f that satisfies constraint: $g(x_0) = 0$.

- If the following regularity condition is satisfied

$$\mathrm{rank}\,(Jg(x_0)) = m,$$

  (i.e the Jacobian matrix has a full rank)
  then there exist Lagrange multipliers $\lambda_1, ..\lambda_m$ of a Lagrangian (2) such that the necessary optimality condition (3) is satisfied:

$$\mathrm{grad}\, F(x_0) = 0$$

Necessary optimality conditions (of 2nd order)

- If rank $(Jg(x_0)) = m$ for $x_0 \in G$ and grad $F(x_0) = 0$ and $HF(x_0)$ is **positive definite** on tangential space

  $$TG(x_0) := \{w \in \mathbb{R}^n : \langle \text{grad } g_i, w \rangle = 0\},$$

  i.e.

  $$w^T HF(x_0) w > 0 \text{ for } w \in TG(x_0 \setminus 0)$$

  then x_0 is a **strict local minimum** of f that satisfies the constraints g.

Remark

- If an admissible set G is **compact** and function f is a **continuous**, then f attains its max/min on G:
  - the stationary point where function has the largest value - global maximum
  - the stationary point where function has the smallest value - global minimum

$\implies$ no need to check the second order optimality condition.

Exercise

Find the global extrema of the function

$$f(x, y, z) = x - 8y + z.$$

on the admissible set defined by $g(x, y, z) = 0$ and $h(x, y, z) = 0$:

$$g(x, y, z) = x^2 + (y + 4)^2 + z^2 - 25$$
$$h(x, y, z) = x^2 + y^2 + z^2 - 9$$

Exercise

Given the function $f : \mathbb{R}^3 \to R$

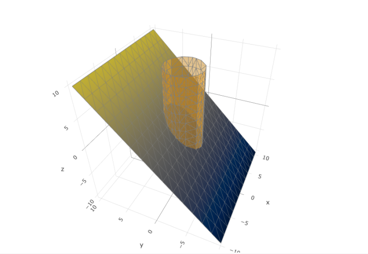
$$f(x, y, z) = z.$$

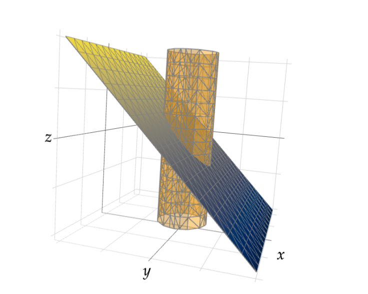
Compute all extrema of the function that satisfy the following constraints

$$g_1(x, y, z) = x^2 + y^2 - 9$$
$$g_1(x, y, z) = y - z$$

and determine whether it is maximum/minimum using the Lagrange multiplier rule.

Exercise





Surface Integrals

Example 1

Integrate the function $f(x, y) = xy$ over the rectangle $[0, 2] \times [1, 4]$.

$$\int_D f(x,y)\,dx = \int_1^4 \int_0^2 x \cdot y\,dx\,dy = \int_1^4 \left(y \cdot \frac{x^2}{2} \right) \Big|_0^2 dy$$

$$= \int_1^4 y \left(\frac{2^2}{2} - \frac{0^2}{2} \right) dy = \int_1^4 2y dy = 4^2 - 1^2 = 15$$

Fubini theorem:

- If f is integrable over D and for all $x \in [a_1, b_1], y \in [a_2, b_2]$ there exist integrals

$$F(x) = \int_{a_2}^{b_2} f(x,y)\,dy \quad G(y) = \int_{a_1}^{b_1} f(x,y)\,dx$$

then it holds

$$\int_D f(x))dx = \int_{a_1}^{b_1}\int_{a_2}^{b_2} f(x,y)\,dy\,dx = \int_{a_2}^{b_2}\int_{a_1}^{b_1} f(x,y)\,dx\,dy$$

Example 2

Integrate the function $f(x, y) = 2 - xy$ over the rectangle $[0, 1] \times [0, 2]$.

$$\int_D f(x, y)\, dx =$$

Example 3

Compute the following integrals:

Some more exercises

Consider the curve

$$f(x, y) = x^4 - x^2 + y^2$$

Find

- symmetries
- tangent lines (horizontal and vertical) for regular points
- singular points and determine their type

Some more exercises

Compute the global extrema of a function

$$f(x, y, z) = xy + z^2$$

subject to constraints

$$g(x, y, z) = x^2 + y^2 - 8 = 0$$

$$h(x, y, z) = x - y + 2z - 2 = 0$$

Thank you!