Analysis III: Auditorium exercise class

Taylor Theorem, Lagrange-remainder, Extrema of Multivariable Functions (max, min, saddle pt.) Implicit Function Theorem

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Zutritt zur Lehrveranstaltung haben nur:

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- RECOVERED
- TESTED

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Thank you for your understanding.
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Let $D \subset \mathbb{R}^n$ be open and convex. Let $f : D \to \mathbb{R}$ be a C^{m+1}–function and $x_0 \in D$. Then the Taylor–expansion in $x \in D$ is well-defined

$$f(x) = T_m(x; x_0) + R_m(x; x_0),$$

Taylor exp. around x_0 of degree m

where

$$R_m(x; x_0) = \sum_{|\alpha|=m+1} \frac{D^\alpha f(x_0 + \theta(x - x_0))}{\alpha!} (x - x_0)^\alpha, \forall \theta \in (0, 1)$$

is a Lagrange–remainder.

$$T_m(x; x_0) = \underbrace{\underbrace{f(x_0)}_{T_0} + \operatorname{grad} f(x_0)(x - x_0)}_{T_1} + \ldots$$

- we approximate f with Taylor polynomial, the more terms we have in T. expansion, the closer we are to f.

  How close — remainder function (error function)

  $(E_m(x;x_0) =)\ R_m(x;x_0) = f(x) - T_m(x;x_0)$

  — how good we fit moving away from x_0

  $T(x_0) = f(x_0) \Rightarrow R(x_0;x_0) = f(x_0) - T_m(x_0;x_0) = 0$

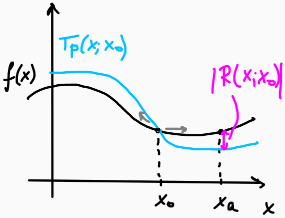
  $R_m^{(m)}(x_0;x_0) = f^{(m)}(x_0) - T_m^{(m)}(x_0;x_0)$

  We choose to use the remainder in a Lagrange form

  $\Rightarrow$ need f to be $(m+1)$-times differentiable

$$R_m^{(m+1)}(x;x_0) = f(x) - T_m^{(m+1)}(x;x_0) \Rightarrow |R_m^{(m+1)}(x;x_0)| = |f^{(m+1)}(x)| \leq M$$

$$\Rightarrow |R_m^{(m+1)}(x_a;x_0)| \leq K$$



Examples

Denote $\xi := x_0 + \Theta(x - x_0)$

$$R_2((x,y);(x_0,y_0)) = \frac{1}{3!}\Big(f_{xxx}(\xi_1;\xi_2)(x-x_0)^3 + 3 f_{xxy}(\xi_1;\xi_2)(x-x_0)^2(y-y_0)$$

$$+ 3 f_{xyy}(\xi_1,\xi_2)(x-x_0)(y-y_0)^2 + f_{yyy}(\xi_1,\xi_2)(y-y_0)^2\Big)$$

$$R_3((x,y);(x_0,y_0)) = \frac{1}{4!}\Big(f_{xxxx}(\xi_1;\xi_2)(x-x_0)^4 + 4 f_{xxxy}(\dots)(x-x_0)^3(y-y_0)$$

$$+ 6 f_{xxyy}(\quad)(x-x_0)^2(y-y_0)^2 + 4 f_{xyyy}(\,)(x-x_0)(y-y_0)^3$$

$$+ f_{yyyy}(\xi_1,\xi_2)(y-y_0)^4$$

Exercise 1

Compute Taylor polynomial $T_2(x; x_0)$ of the function $f(x, y, z) = xe^z - y^2$ centered around a point $x_0 = (1, -1, 0)^T$.

$$T_2(x; x_0) = f(x_0) + \nabla f(x_0)(x - x_0) + \frac{1}{2}(x - x_0)^T Hf(x_0)(x - x_0)$$

$$\nabla f(x, y, z) = \begin{pmatrix} e^z \\ 2y \\ xe^z \end{pmatrix} \qquad Hf(x, y, z) = \begin{pmatrix} 0 & 0 & e^z \\ 0 & -2 & 0 \\ e^z & 0 & xe^z \end{pmatrix}$$

$$\nabla f(1; -1; 0) = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} \qquad Hf(1; -1; 0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \qquad f(x_0) = 0$$

$$\begin{aligned} T_2(x; x_0) &= 0 + (x - x_0) + 2(y - y_0) + (z - z_0) \\ &\quad + \frac{1}{2}\left(-2(y - y_0)^2 + (z - z_0)^2 + 2(x - x_0)(z - z_0)\right) \\ &= (x - 1) + 2(y + 1) + z - (y + 1)^2 + \frac{1}{2}z^2 + (x - 1)z \end{aligned}$$

The estimate on the remainder

The Taylor approximation of a function reads as

all partial derivatives of order m+1

$$f(x) = T_m(x; x_0) + O(\| x - x_0 \|^{m+1})$$

If $D^\alpha f, |\alpha| = m + 1$ are bounded by $C > 0$ in a neighborhood of x_0 then the estimate holds

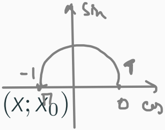
$$|R_m(x_0; x)| \leq \frac{n^{m+1}}{(m+1)!} C \| x - x_0 \|_\infty^{m+1}$$

n – # of variables of f ($f : \mathbb{R}^n \to \mathbb{R}$)
m – degree of Taylor polynomial ($T_m(x; x_0)$)

$\|\bar{v}\|_\infty = \max\{ |v_i|, i = \overline{1,n} \}$
$v = (1; -5; 6)$ $\|v\| = \max\{|1|, |-5|, |6|\} = 6.$

Exercise 3

$x \in [0;\pi]$: $\cos x \in [-1;1]$
$\sin x \in [0;1]$



Compute $T_2(x;x_0)$ of a function $f(x,y) = e^x \cos(y)$ at the point $x_0 = (0,0)^T$ and the estimate for the associated remainder $R_2(x;x_0)$ for $(x,y) \in [-2,2] \times \cancel{[-2,2]}$. $[0;\pi]$

$$\nabla f(x) = \begin{pmatrix} e^x \cos y \\ -e^x \sin y \end{pmatrix} \qquad Hf(x) = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ -e^x \sin y & -e^x \cos y \end{pmatrix} \qquad \nabla f(x_0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$Hf(x_0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Plug into formula for $T_2(x;x_0) = \ldots$

To compute the estimate on remainder: bound $\forall D^{(m+1)} f$ from above:

$m+1 = 3$ $\quad (f: D \subset \mathbb{R}^2 \to \mathbb{R})$

$|f_{xxx}| = |e^x \cos y| \le e^2 \cdot 1 \qquad |f_{xxy}| = |e^x \sin y| \le e^2 \cdot 1$

$|f_{xyy}| = |e^x \cos y| \le e^2 \cdot 1 \qquad |f_{yyy}| = |e^x \sin y| \le e^2 \cdot 1$

Choose the largest found as C:
$C = e^2$

$$|R_2(x_0;x)| \le \frac{2^{2+1}}{3!} \cdot e^2 \max\{|x-0|, |y-0|\}^3 = \frac{2^3}{3!} \cdot e^2 \max\{|x|,|y|\}^3$$

Compute $T_2(x; x_0)$ of a function $f(x, y) = \cos(x^2 + y^2)$ at the point $x_0 = (0, 0)^T$ and the approximation error for $(x, y) \in [0, \frac{\pi}{4}] \times [0, \frac{\pi}{4}]$.

$f(x_0) = 1$; $\nabla f(x) = \begin{pmatrix} -2x\sin(x^2+y^2) \\ -2y\sin(x^2+y^2) \end{pmatrix}$ $\quad \nabla f(x_0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$Hf(x) = \begin{pmatrix} -2\sin(x^2+y^2) - 4x^2\cos(x^2+y^2) & -4xy\cos(x^2+y^2) \\ -4xy\cos(x^2+y^2) & -2\sin(x^2+y^2) - 4y^2\cos(x^2+y^2) \end{pmatrix}$$

$Hf(x_0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $\Rightarrow$ $T_2((x,y);(0,0)) = f(0,0) + f_x(0,0) \cdot x + f_y(0,0)y + \frac{1}{2}\left(f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + f_{yy}y^2 \right) = 1$

For the remainder need to check 3rd order derivatives:

$x, y \in [0; \frac{\pi}{4}]$
$\Rightarrow x^2+y^2 \in [0; \frac{\pi^2}{16}]$

$f_{xxx} = -12x\cos(x^2+y^2) + 8x^3\sin(x^2+y^2)$

$f_{xxy} = -4y\cos(x^2+y^2) + 8x^2y\sin(x^2+y^2)$

$$f_{xyy} = -4x\cos(x^2+y^2) + 8y^2x\sin(x^2+y^2)$$
$$f_{yyy} = -12y\cos(x^2+y^2) + 8y^3\sin(x^2+y^2)$$

$$\left.\begin{aligned}&\Theta\in(0,1)\\ &\xi := x_0+\Theta x\end{aligned}\right\} \Rightarrow \boxed{(\xi_1,\xi_2)\in\left(0;\tfrac{\pi}{4}\right)\times\left(0;\tfrac{\pi}{4}\right)}$$

$$|R_2(x,y;0,0)| = \frac{1}{3!}\Big(\Big| f_{xxx}(\xi_1,\xi_2)x^3 + 3f_{xxy}(\xi_1,\xi_2)x^2y + 3f_{xyy}(\xi_1,\xi_2)xy^2 + f_{yyy}(\xi_1,\xi_2)y^3\Big|\Big)$$

$$\overset{\triangle\text{-ineq.}}{\leq} \frac{1}{3!}\Big(|f_{xxx}(\xi_1,\xi_2)|\cdot|x|^3 + 3|f_{xxy}||x^2y| + 3|f_{xyy}||xy^2| + |f_{yyy}||y^3|\Big)$$

$|\sin t| \leq 1$; $|\cos t| \leq 1$

$$\Rightarrow |f_{xxx}||x|^3 = \Big|-12\xi_1\cos(\xi_1^2+\xi_2^2) + 8\xi_1^3\sin(\xi_1^2+\xi_2^2)\Big||x|^3$$
$$\leq \Big(|12\xi_1|\cdot\underbrace{|\cos(\xi_1^2+\xi_2^2)|}_{\leq 1} + 8|\xi_1|^3\underbrace{|\sin(\xi_1^2+\xi_2^2)|}_{\leq 1}\Big)\cdot|x|^3$$
$$\leq \Big(12\cdot\frac{\pi}{4} + 8\Big(\frac{\pi}{4}\Big)^3\Big)\Big(\frac{\pi}{4}\Big)^3$$

analogously:

$$3|f_{xxy}||x^2y| \le 3\left(4\cdot\frac{\pi}{4}+8\cdot\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$3|f_{xyy}|\cdot|xy^2| \le 3\left(4\cdot\frac{\pi}{4}+8\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$|f_{yyy}||y^3| \le \left(12\cdot\frac{\pi}{4}+8\left(\frac{\pi}{4}\right)^3\right)\left(\frac{\pi}{4}\right)^3$$

$$\Rightarrow \left|f(x,y)-T_2((x,y);(0,0))\right| \le \frac{\pi^3}{3!\,4^3}\left(48\cdot\frac{\pi}{4}+64\left(\frac{\pi}{4}\right)^3\right) \simeq 5.5476$$

Max error for $x=y=\frac{\pi}{4}$:

$$\left|f\left(\frac{\pi}{4},\frac{\pi}{4}\right)-T_2\left(\frac{\pi}{4},\frac{\pi}{4};0,0\right)\right| = \left|\cos\left(2\cdot\frac{\pi^2}{4^2}\right)-1\right| = 0.6692$$

Extrema of multivariable function

Let $D \subset \mathbb{R}^n, f : D \to \mathbb{R}$ and $x_0 \in D$. Then at x_0 the function f has

- a (strict) global maximum if $\forall x \in D : f(x) \overset{(<)}{\leq} f(x_0)$

- a (strict) local maximum if

$$\exists \epsilon > 0 \; \forall x \in D \text{ with } \| x - x_0 \| < \epsilon : f(x) \overset{(<)}{\leq} f(x_0)$$

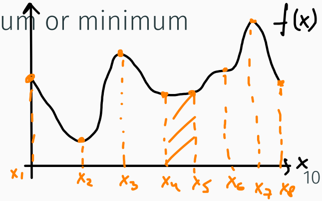
// $f(x_0)$ is larger than $f(x)$ in the $B_\varepsilon(x_0)$ $\forall x$.

- analogously for minima

Note: x_0 is called an extremum if it is maximum or minimum

x_1 - strict local min
x_2 - str. global min
x_3 - strict local max
$x \in [x_4, x_5]$ - local min

x_6 - saddle pt
x_7 - strict global max
x_8 - strict local min

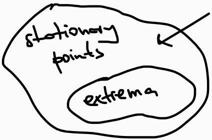


Stationary points

- The points $x_0 \in D$ for which it holds

$$\operatorname{grad} f(x_0) = 0$$

  are called stationary points (critical points) of f.
- Stationary points are not necessarily extrema.



saddle pts,

Necessary optimality conditions

necessary – it should hold for x_0 to be a candidate for an extrema, but we'll still have to do additional checks later

"A" => "B" "B" should be true for A to be true (but A still may be false)

1st order // x_0 - extremum => x_0 is a stationary

- Let $f \in D$ is $C^1, x_0 \in D$ - local extremum $\implies$ grad $f(x_0) = 0$
- 2nd order Let $f \in D$ is $C^2, x_0 \in D$ - stationary point
    - if x_0 is local min (max)

      $\implies$ $H\,f(x_0)$ positive (negative) semi-definite.

$Hf(x_0) \geqslant 0$ – pos. semi-def

$Hf(x_0) \leqslant 0$ – negative-def.

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix}$$

Sylvester's criterion

$$A > 0 \text{ if } \begin{cases} a_1 > 0 \\ \det\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix} > 0 \\ \det(A) > 0 \end{cases}$$

$$A < 0 : \begin{cases} a_1 < 0 \\ \det\begin{pmatrix} a_1 & a_2 \\ a_4 & a_5 \end{pmatrix} > 0 \\ \det(A) < 0 \end{cases}$$

if some are 0s => semi-def

if none of the above => indefinite

$+++ \Rightarrow A \succ 0$

$-+- \quad A \prec 0$

Example

$+0+ \Rightarrow A \succcurlyeq 0$ – pos. semi-def

$-0- \Rightarrow A \preccurlyeq 0$

$+-- \Rightarrow$ indefinite

Analogously – using eigenvalues (more at the exercise class)

$\forall \lambda > 0 \to$ min

$\forall \lambda < 0 \to$ max

$\forall \lambda \geq 0 \quad \exists \lambda \neq 0 \to$ min / saddle pt.

$\forall \lambda \leq 0 \quad \exists \lambda \neq 0 \to$ max / saddle

$\exists \lambda > 0, \exists \lambda < 0 \to$ saddle

Sufficient optimality conditions

"A" $\Leftarrow$ "B" If "B" is true, then "A" is true.
The truth of B guarantees the truth of A.

- Let $f \in D$ is $C^2, x_0 \in D$ - stationary point
  - $H\,f(x_0)$ positive (negative) definite $\implies$ x_0 is strict local min (max)
  - $H\,f(x_0)$ indefinite $\implies$ x_0 is a saddle point

Examples

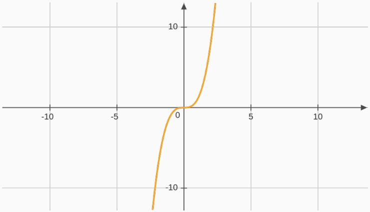


Figure 1: $f(x) = x^3, f'(0) = 0$ but $x^* = 0$ isn't extremum

x^* is a stationary pt, but is not extremum.

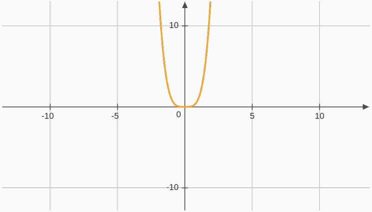


Figure 2: $f(x) = x^4, f''(x) = 12x^2, f''(0) = 0$ but still $x^* = 0$ is minimum

$f''(0) = 0 \geq 0$ – so 2nd order suff. cond is not satisfied but x^* is still a minimum.

Exercise 5

Compute the stationary points of the following functions and determine whether it is min/max/saddle point

1. $f(x, y) = xy + x - 2y - 2$
2. $f(x, y) = 2x^3 - 3xy + 2y^3 - 3$
3. $f(x, y) = (x^2 + 2y^2)e^{-x^2-y^2}$
4. $f(x, y) = x^5 - 3x^3 + y^2 + 15,$

in the exercise class

1. $f(x,y) = xy + x - 2y - 2$

$\text{grad } f(x,y) = (y+1,\ x-2)$

To compute stationary points:

$\text{grad } f(x,y) = 0$: $\begin{cases} y+1=0 \\ x-2=0 \end{cases} \Rightarrow \begin{matrix} y=-1 \\ x=2 \end{matrix}$

$\Rightarrow$ have only one stationary pt. $(2;-1)$

To check what it is check second-order opt. condition:

$Hf(x,y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ $\quad Hf(2;-1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: A$

$a_1: 0$

$\det Hf = -1 < 0 \Rightarrow$ not pos. def, not neg. def $\Rightarrow$ indef. $\Rightarrow$ saddle point

$0 \ - \ \Rightarrow$

Alternatively: using eigenvalues: $\det(A - \lambda I) = 0$

$\det \begin{pmatrix} -\lambda & 1 \\ 1 & -\lambda \end{pmatrix} = 0 \Rightarrow \lambda^2 - 1 = 0$

$\Rightarrow \lambda = \pm 1$

$\lambda_1 > 0$; $\lambda_2 < 0 \Rightarrow$ saddle

2. $f(x,y) = 2x^3 - 3xy + 2y^3 - 3$

$\text{grad } f(x,y) = (6x^2 - 3y \,;\, -3x + 6y^2)$

$\begin{cases} 6x^2 - 3y = 0 \\ -3x + 6y^2 = 0 \end{cases}$ $\Rightarrow 2x^2 = y$, $\Rightarrow -3x + 24x^4 = 0$ $\quad \begin{cases} 2x^2 = y \\ 3x(8x^3 - 1) = 0 \end{cases}$

$\begin{cases} \left[\begin{array}{l} x = 0 \\ 8x^3 - 1 = 0 \end{array}\right. \\ 2x^2 = y \end{cases}$ $\quad \left[\begin{array}{l} \begin{cases} x = 0 \\ y = 0 \end{cases} \\ \begin{cases} x = 1/2 \\ y = 1/2 \end{cases} \end{array}\right.$ $\Rightarrow$ two stationary pts: $(0,0)$; $(\frac{1}{2}; \frac{1}{2})$

$Hf(x,y) = \begin{pmatrix} 12x & -3 \\ -3 & 12y \end{pmatrix}$ $\quad Hf(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$ $\quad 0$

$\det Hf(0,0) = -9 < 0$

$\Rightarrow$ saddle

$Hf(\frac{1}{2}; \frac{1}{2}) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$ $\quad 6 > 0$

$36 - 9 = 27 > 0$ $\Rightarrow$ pos. def $\Rightarrow$ minimum.

alt via eigs: $\det\begin{pmatrix} 6-\lambda & -3 \\ -3 & 6-\lambda \end{pmatrix} = 0$ $\quad (6-\lambda)^2 = 9$

$6 - \lambda = \pm 3$

$\lambda = 6 \mp 3$;

$\lambda_1 = 3 > 0 \quad \lambda_2 = 9 > 0$

$\Rightarrow$ min.

Implicitly Defined Functionss

Consider a system of nonlinear equations

$$g(x) = 0,$$

$$g\begin{pmatrix}x_1\\x_2\\\vdots\\x_n\end{pmatrix}=\begin{pmatrix}y_1\\\vdots\\y_m\end{pmatrix}$$

with $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$, $m < n$, i.e more unknowns than equations. - underdetermined system of equations.

We want to solve such systems locally expressing some variables via other.

Implicit Function Theorem

Let $g : D \subset \mathbb{R}^n \to \mathbb{R}^m$ be a C^1 - function. Let $(x, y) \in D$, where $x \in \mathbb{R}^{n-m}, y \in \mathbb{R}^m$. Let $(x_0, y_0) \in D$ - solution to $g(x_0, y_0) = 0$. If the Jacobian matrix

$$\frac{\partial g}{\partial y}(x_0, y_0) := \begin{pmatrix} \frac{\partial g_1}{\partial y_1} & \dots & \frac{\partial g_1}{\partial y_m} \\ \dots & & \dots \\ \frac{\partial g_m}{\partial y_1} & \dots & \frac{\partial g_m}{\partial y_m} \end{pmatrix}$$

is regular, then there exist neighbourhoods U of x_0, V of $y_0, U \times V \subset D$ and a uniquely determined continuous differentiable function $f : U \to V : f(x_0) = y_0$ and $g(x, f(x)) = 0$ for all $x \in U$ and

$$J\,f(x) = -\left(\frac{\partial g}{\partial y}(x, f(x))\right)^{-1}\left(\frac{\partial g}{\partial x}(x, f(x))\right)$$

Exercise 7

Can the equation $(x^2 + y^2 + 2z^2)^{\frac{1}{2}} = cos(z)$ be solved uniquely for y in terms of x, z near $(0, 1, 0)$? For z in terms of x and y?

$(x^2+y^2+2z^2)^{1/2} - \cos(z)$

$F(x, y, z) = (0+1+0)^{1/2} - 1 = 0$

$F(0, 1, 0) = \frac{1}{\cancel{2}}(x^2+y^2+2z^2)^{-1/2} \cdot \cancel{2}y \Big|_{(0,1,0)} = \frac{y}{\sqrt{x^2+y^2+2z^2}} \cdot \frac{1}{\sqrt{0+1+0}} = 1 \neq 0$

$\Rightarrow$ from implicit f. theorem we can solve for y in terms of (x, z)

$\frac{\partial F}{\partial y}(0, 1, 0) = \left(\frac{1}{\cancel{2}}(x^2+y^2+2z^2)^{-1/2} \cdot \cancel{2} \cdot 2z + \sin z\right)\Big|_{(0,1,0)}$

$\frac{\partial F}{\partial z}(0, 1, 0) = 0 + \sin 0 = 0$

$\Rightarrow F(x, y, z)$ cannot say whether F can be expressed as $z = f(x, y)$

Exercise 8

Consider the function $F(x, y, z, u, v) : \mathbb{R}^5 \to \mathbb{R}^2$ given by

$$F(x, y, z, u, v) = \begin{pmatrix} xy^2 + xzu + yv^2 - 3 \\ u^3yz + 2xv - u^2v^2 - 2 \end{pmatrix}$$

Can we solve for u, v as functions of x, y, z near $(1, 1, 1, 1, 1)$?

Notice that F(1,1,1,1,1) = 0.

$$\begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} = \begin{pmatrix} xz & 2yv \\ 3u^2yz - 2uv^2 & 2x - 2u^2v \end{pmatrix}$$

$$A := \begin{pmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{pmatrix} (1, 1, 1, 1, 1) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

$\det A = 1 \cdot 0 - 2 = -2 \neq 0$

$\Rightarrow$ we may solve for u, v in terms of (x, y, z) locally near $(1,1,1,1,1)$

// $\exists\ (u,v) = f(x,y,z)$ that solves F

Some more exercises

Compute $T_2(x; x_0)$ of a function

$$f(x, y) = \cos(x)\sin(y)e^{x-y}$$

at the point $x_0 = (0, 0)^T$ and the associated remainder $R_2(x; x_0)$

Some more exercises

Compute Taylor polynomial of second degree $T_2(x; x_0)$ of a function

$$f(x, y) = \sin(x + y) + ye^{x-y}$$

at the point $x_0 = (0, 0)^T$ and the estimate for the Lagrange remainder for $|x| \leq 0.1, |y| \leq 0.1.$

Thank you!