

Analysis III: Auditorium exercise class

Contour lines, gradients, higher-order derivatives

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BITTE BEACHTEN SIE DIE 3G-REGEL! PLEASE OBEY THE 3G RULE!



Zutritt zur Lehrveranstaltung
haben nur:

- VOLLSTÄNDIG GEIMPFT
- GENESENE
- GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen
können, müssen Sie bitte den Raum
jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted
to persons who are:

- FULLY VACCINATED
- RECOVERED
- TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this,
please leave the room now.
Otherwise you could be banned from
the room!

Thank you for your understanding.
Protect yourself and others!

Organisation

- office hours
 - Mon 13:15-14:15 bi-weekly (18.10., 01.11., 15.11. ...)
 - Location: SBC 3-E, Room 4.012
 - or in BBB
 - appointment by email sofiya.onyshkevych@uni-hamburg.de
- Room Change: Mon, 4pm - N0009. Last class - A0.18.

website

Setting

Given: $f : D \rightarrow \mathbb{R}, D \subset \mathbb{R}^n$

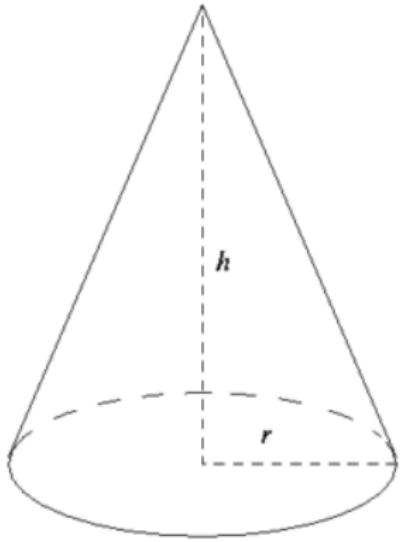
$$f: \underbrace{(x_1, x_2, \dots, x_n)}_{\in D} \mapsto x^* \in \mathbb{R}$$

Example of a fun of mult. var.

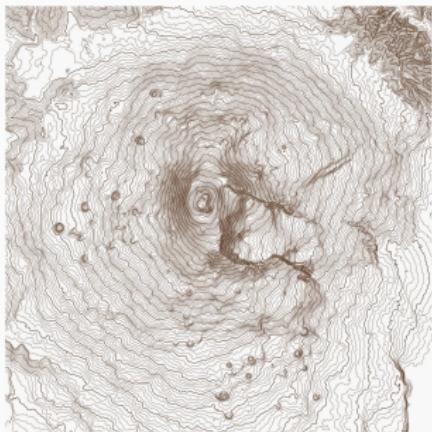
$$V(h, r) = \frac{1}{3}\pi r^2 h$$

$$D := \{(h, r) \in \mathbb{R}^2 : h > 0, r > 0\}$$

domain



What is a level curve?



(a) A relief image of Mount Etna.
Source: Wikipedia

"level curves" or "contour lines"
level set
 $\text{Gren } z = f(x,y). \quad f(x,y) = C$



(b) A perspective photo of Etna

Example 1

Circle in 2D : $(x - x_0)^2 + (y - y_0)^2 = r^2$
 $C(x_0, y_0); r$

Plot contour lines of the given function

$$z = f(x, y) = x^2 + 2y^2 + 1.$$

Solution:

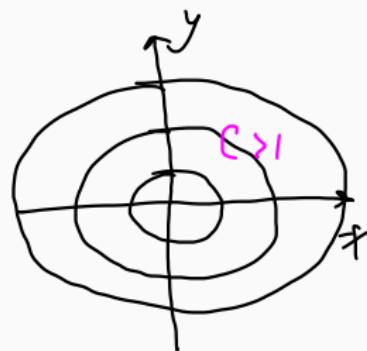
$$x^2 + 2y^2 + 1 = C$$

$$x^2 + 2y^2 = C - 1$$

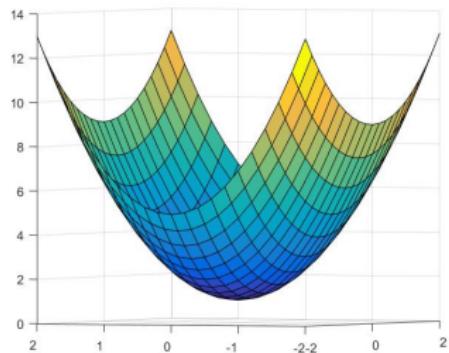
$$C = 1 : \quad x^2 + 2y^2 = 0 \text{ - point } (0,0)$$

$C > 1 :$ ellipse

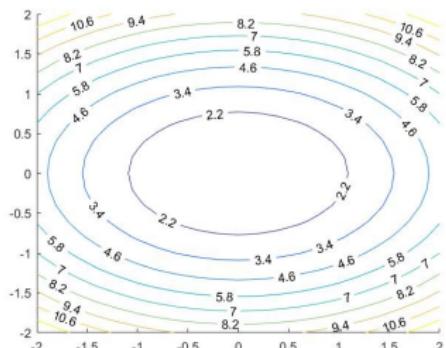
$C < 1 :$ no lines in a real space



Example 1



(a) Surf plot of f .



(b) Contour plot of f .

Code explanation on later slides

Exercise 1

Plot contour lines of the given function

$$z = f(x, y) = x + y^2 + 1$$

Solution:

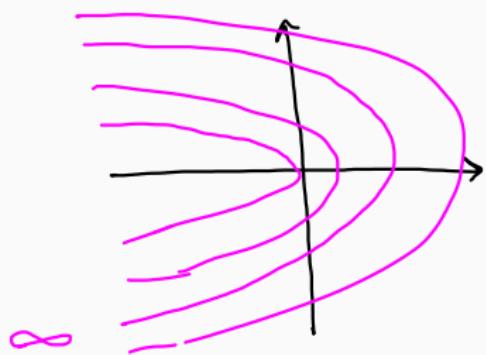
$$x + y^2 + 1 = c$$

$$x = -y^2 + c - 1$$

parabola

Translation dep. on c

Domain - \mathbb{R}^2

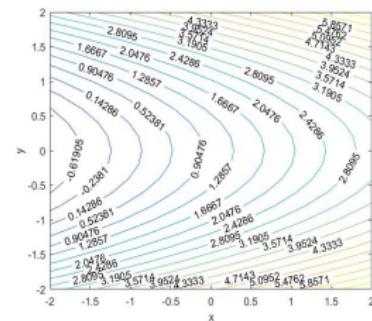


Exercise 2

Plot contour lines of the given function and determine the direction of the gradient

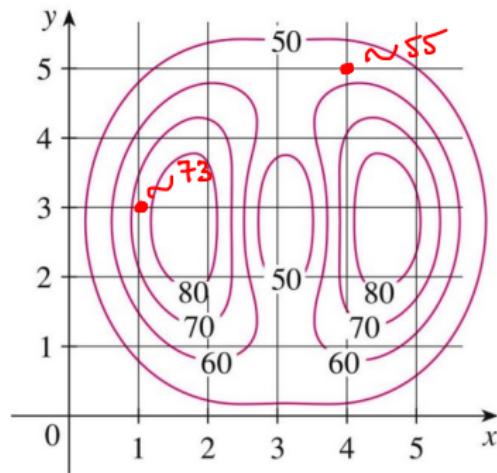
$$z = f(x, y) = x + y^2 + 1$$

Solution:



Exercise 3

Given the contour map¹ for a function f , estimate the values of $f(1, 3)$ and $f(4, 5)$. What can you say about the shape of the graph of f ?



¹<https://www.usna.edu/Users/math/uhan/sm223/lessons/12%20Level%20Curves.pdf>

Plotting in Matlab

```
x=[-2 : .2 : 2]; % Interval [-2;2]; step 0.2
```

```
y=[-2 : .2 : 2];
```

```
[X,Y] = meshgrid(x,y);
```

$Z=X.^2 + 2*Y.^2 + 1;$ % $Z=x^2+2y^2+1$. Be careful with pointwise operations.

```
mesh(X,Y,Z)
```

```
contour(X,Y,Z,30) % plotting contour lines  
    ↑ amount of lines
```

```
xlabel('x axis'); % labeling the axes
```

```
[C,h] = contour(X,Y,Z,30);  
clabel(C,h);
```

} to have values of contour lines displayed on the graph

alternative:
 $\text{linspace}(a,b,n);$
[a,b], n points

Continuity and Differentiability

Let $D \subset \mathbb{R}^n$ —open, $f: D \rightarrow \mathbb{R}$, $x_0 \in D$.

f is called **partially differentiable in x_0 with respect to x_i** if the limit

$$f_{x_i}(x_0) = \frac{\partial f}{\partial x_i}(x_0) := \lim_{t \rightarrow 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

exists. Here e_i denotes the i —th unit vector. This limit is called **partial derivative** of f with respect to x_i at x_0 .

f is (continuously) **partially differentiable** if f is (continuously) partially differentiable w.r.t. each component x_1, \dots, x_n .

Examples

Are functions $f_1 - f_3$ partially differentiable? Compute the partial derivatives of the given functions.

✓ $\cdot f_1(x_1, x_2) = 3x_1^3 + 8x_2^2$

$$\frac{\partial f_1}{\partial x_1} = 3 \cdot 3x_1^2$$

$$\frac{\partial f_1}{\partial x_2} = 8 \cdot 2x_2$$

✗ $\cdot f_2(x_1, x_2) = x_1^4 - |x_2|$

$$\frac{\partial f_2}{\partial x_1} = 4x_1^3$$

✓ $\cdot f_3(x_1, x_2) = 3x_1 - 5x_2$

$$\frac{\partial f_3}{\partial x_1} = 3$$

$$\frac{\partial f_3}{\partial x_2} = -5$$

Gradient and nabla-operator

Let $D \subset \mathbb{R}^n$ – open, $f: D \rightarrow \mathbb{R}$ partially differentiable.

- We denote the row vector

$$\underbrace{\text{grad } f(x_0)} := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right) = (f_{x_1}, f_{x_2}, \dots, f_{x_n})$$

as **gradient** of f at x_0 .

- We denote the symbolic vector

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^T$$

as **nabla-operator**

- Thus we obtain the **column vector**

$$\underbrace{\nabla f(x_0)} := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right)^T$$

Examples

$$1 \quad f_1(x,y) = x^3 + xy + y^4$$

$$f_x = 3x^2 + y \quad f_y = x + 4y^3$$

$$\text{grad } f_1(x,y) = (3x^2 + y, x + 4y^3)$$

Compute the gradient of the given functions

$$2 \quad f_2(x,y) = x^4 \ln y - y^5 e^x \quad \frac{\partial f_2}{\partial x}(x,y) = 4x^3 \ln y - y^5 e^x$$

$$\frac{\partial f_2}{\partial y}(x,y) = \frac{x^4}{y} - 5y^4 e^x$$

$$\text{grad } f_2(x,y) = \left(4x^3 \ln y - y^5 e^x, \frac{x^4}{y} - 5y^4 e^x \right)$$

More Examples

Compute the gradient of the given function

$$f(x, y, z) = x^2 y \cos(z)$$

$$f_x(x, y, z) = 2xy \cos(z)$$

$$f_y(x, y, z) = x^2 \cos(z)$$

$$f_z(x, y, z) = -\sin(z)x^2y$$

$$\text{grad } f(x) = (f_{x_1}(x), \dots, f_{x_n}(x)) = (2xy \cos(z), x^2 \cos(z), -x^2y \sin(z))$$

Tangent Planes

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S , and let C be any curve passing through P_0 and lying entirely in S . If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0 .

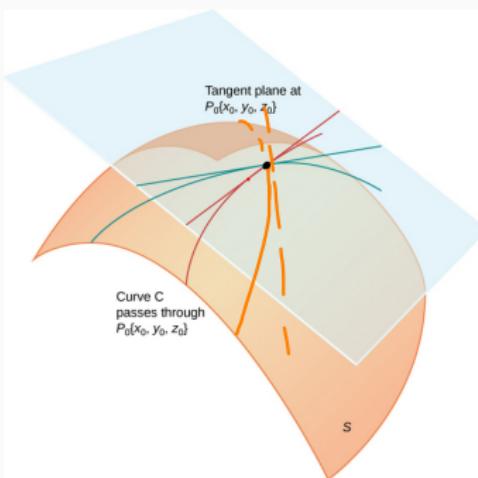
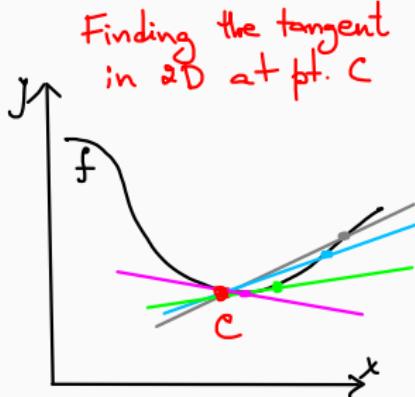


Figure 3: The tangent plane to a surface S at a point P_0

Exercise

Compute the tangent plane to the graph f at point $(x_0, y_0) := (1, 3)$.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

Solution:

$$T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(1, 3) = 4 - 1 - 9 = -6$$

$$f_x(x, y) = -2x$$

$$f_y(x, y) = -2y$$

$$\begin{aligned} T &= -6 - 2x(x-1) - 2y(y-3) = -6 - 2x^2 + 2x - 2y^2 + 6y \\ &= -2x^2 - 2y^2 + 2x + 6y - 6 \end{aligned}$$

Exercise

$$f(x,y) = x^2 + y^2$$

Compute the tangent plane to the surface $z = x^2 + y^2$ at the point $(x_0, y_0, z_0) := (1, 2, 5)$.

$$f_0 = f(x_0, y_0) = f(1, 2) = 5$$

Solution:

$$T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f_x(x, y) = 2x \quad f_y(x, y) = 2y$$

$$f_x(1, 2) = 2 \quad f_y(1, 2) = 4$$

$$\begin{aligned} T(x, y) &= 5 + 2(x - 1) + 4(y - 2) \\ &= 5 + 2x - 2 + 4y - 8 = 2x + 4y - 5 \end{aligned}$$

Exercise 2 (part 2)

Theorem The gradient is orthogonal to the contour lines.

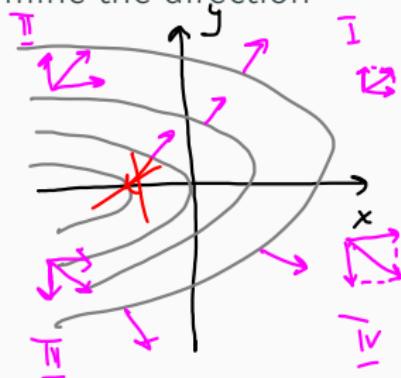
Plot contour lines of the given function and determine the direction of the gradient

$$z = f(x, y) = x + y^2 + 1$$

Solution:

$$\text{grad}(f(x,y)) = \begin{pmatrix} 1, 2y \end{pmatrix} \quad D = \mathbb{R}^2$$

$\text{I: } (+,+)$ $\text{II: } (+,-)$
 $\text{III: } (+,+)$ $\text{IV: } (+,-)$



Higher-order derivatives

Compute the derivatives of the second order of the given function

$$f(x, y, z) = x^2 y \cos(z)$$

$$\begin{pmatrix} f_x(x, y, z) \\ f_y(x, y, z) \\ f_z(x, y, z) \end{pmatrix}^T = \begin{pmatrix} y^2 \cos(z) \\ 2xy \cos(z) \\ xy^2 (-\sin(z)) \end{pmatrix}$$

$$f_{yx} = \frac{\partial}{\partial x} f_y(x, y, z) = \frac{\partial^2 f}{\partial x \partial y} = 2y \cos z$$

$$\begin{matrix} f_{xy} \\ f_{xx} \\ f_{zy} \\ f_{zz} \end{matrix}$$

$$\frac{\partial}{\partial y} f_y(x, y, z) = f_{yy} = \frac{\partial^2 f}{\partial y^2} = 2x \cos z$$

$$\frac{\partial}{\partial z} f_y(x, y, z) = -2xy \sin z$$

Hessian Matrix

If all second partial derivatives of f exist and are continuous over the domain of the function, then **the Hessian matrix H** of f is a square $n \times n$ matrix:

$$H_f(x) = \text{Hess } f(x) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix}(x) = \begin{pmatrix} f_{x_1 x_1}, \dots, f_{x_1 x_n} \\ f_{x_n x_1}, \dots, f_{x_n x_n} \end{pmatrix}$$

$$(H_f)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}$$

\Rightarrow $f_{x_i x_j} = f_{x_j x_i}$

f is 2-times cont. differentiable

Exercise

Compute first and second derivatives of f

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y, z) = z^2 + \ln(xy^2)$$

$$\text{Hf}(x, y, z) = \begin{pmatrix} -\frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{2}{y^3} & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$f_x = y^2 \frac{1}{xy^2} = \frac{1}{x} = x^{-1}$$

$$f_{xx} = -x^{-2} \quad f_{xy} = 0 \quad f_{xz} = 0$$

$$f_y = \frac{x \cdot 2y}{xy^2} = \frac{2}{y} = 2y^{-1}$$

$$f_{yx} = 0 \quad f_{yy} = \frac{-2}{y^2} \quad f_{yz} = 0$$

$$f_z = 2z$$

$$f_{zx} = 0 \quad f_{zy} = 0 \quad f_{zz} = 2$$

Exercise

Compute the third order partial derivative f_{xyy} of a function f :

$$f(x, y) = \ln(xy^2 + 2y)$$

$$f_x = \frac{1}{xy^2 + 2y} \cdot y^2 = \frac{y}{xy + 2}$$

$$f_{xy} = \frac{xy^2 - xy}{(xy + 2)^2} = \frac{2}{(xy + 2)^2} = 2(xy + 2)^{-2}$$

$$f_{xyy} = 2 \cdot (-2) (xy + 2)^{-3} \cdot x = -\frac{4x}{(xy + 2)^3}$$

Exercise

Compute the first and second order partial derivatives of a function f :

$$f(x, y) = 8x - 2x^2y^2$$

$$f_x^{(x,y)} = 8 - 4xy^2$$

$$f_{xx} = -4y^2$$

$$f_{yx} = f_{xy} = -8xy$$

$$f_y^{(x,y)} = -2x^2 \cdot 2y = -4x^2y$$

$$f_{xy} = -4x \cdot 2y = -8xy$$

$$f_{yy} = -4x^2$$

$$\mathbf{H}(f(x,y)) = \begin{pmatrix} -4y^2 & -8xy \\ -8xy & -4x^2 \end{pmatrix}$$

Exercise

Compute the first and second order partial derivatives of a function f for $x \neq 0$:

$$f(x, y, z) = e^z + \frac{1}{x} + xe^{-y}$$
$$f_x(x, y, z) = (-1)x^{-1-1} + e^{-y} = -x^{-1} + e^{-y}$$
$$f_y(x, y, z) = xe^{-y}(-1) = -xe^{-y}$$
$$f_z(x, y, z) = e^z$$
$$f_{xx} = (-1)(-2)x^{-3} = 2x^{-3}$$
$$f_{xy} = -e^{-y}$$
$$f_{xz} = 0$$
$$f_{yy} = +xe^{-y}$$
$$f_{yz} = f_{xy} = -e^{-y}$$
$$f_{yz} = 0$$
$$f_{zx} = 0 \quad f_{zy} = 0 \quad f_{zz} = e^z$$

$$Mf(x, y) = \begin{pmatrix} 2x^{-3} & -e^{-y} & 0 \\ -e^{-y} & xe^{-y} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Some more exercises

Compute partial derivatives of given functions

- $(3x - 5y)^4 \quad \text{grad } f(x,y) = \left(4(3x-5y)^3 \cdot 3 ; \ 4(3x-5y)^3(-5) \right)$
- $\frac{\sin(xyz)}{x^2}$
- $\frac{y}{x} e^{-(x^2+y^2)}$
- $\arctan \frac{x}{y} = \left(\frac{1}{1 + (\frac{x}{y})^2} \cdot \frac{1}{y} ; \ \frac{1}{1 + (\frac{x}{y})^2} \cdot \frac{x}{y^2} \cdot (-1) \right)$
 $x y^{-1}$

Exercise

Compute the tangent plane to the surface defined by the function $f(x, y) = \sin(2x) \cos(3y)$ at the point $(\pi/3, \pi/4)$. Solution:

$$T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f_x(x_0, y_0) = 2 \cos(2x_0) \cos(3y_0) \quad f_y(x_0, y_0) = -3 \sin(2x_0) \sin(3y_0)$$

$$f\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = \sin\left(\frac{2\pi}{3}\right) \cos\left(3\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} \left(-\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{6}}{4}$$

$$f_x\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = 2 \cos\left(2\frac{\pi}{3}\right) \cos\left(3\frac{\pi}{4}\right) = 2\left(-\frac{1}{2}\right)\left(-\frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}}{2}$$

$$f_y\left(\frac{\pi}{3}, \frac{\pi}{4}\right) = -3 \sin\left(2\frac{\pi}{3}\right) \sin\left(3\frac{\pi}{4}\right) = -3\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) = -\frac{3\sqrt{6}}{4}$$

$$T = -\frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{2} \left(x - \frac{\pi}{3}\right) - \frac{3\sqrt{6}}{4} \left(y - \frac{\pi}{4}\right)$$

Thank you!

Draw contour lines and determine the direction of the gradient of $f(x,y) = \frac{1}{x} \ln y$.

Solution. as usual $\frac{1}{x} \ln y = C$

Attention! For what C it makes sense?

$D := \{x \neq 0, y > 0\}$ - domain

$$\ln y = Cx$$

$$y = e^{Cx}, x \neq 0$$

$$C=0, y=1$$

$$C=1, y=e^x$$

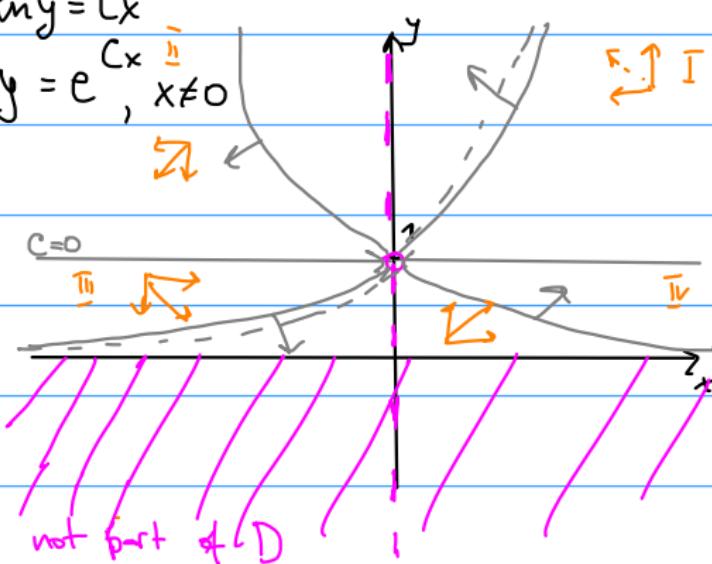
$$C=-1, y=e^{-x}$$

$$\text{I: } x \in (0, +\infty), y \in (1, +\infty)$$

$$\text{II: } x \in (-\infty, 0), y \in (1, +\infty)$$

$$\text{III: } x \in (-\infty, 0), y \in (0, 1)$$

$$\text{IV: } x \in (0, +\infty), y \in (0, 1)$$



Gradient is orthogonal to the level line \Rightarrow determine from plot
but the direction?

$$\nabla f = \left(-\frac{1}{x^2} \ln y, \frac{1}{xy} \right)^T$$

I. $y > 1 \Rightarrow \ln y > 0 \quad \nabla f(-, +)$

$$x > 0$$

II. $\ln y > 0; x < 0 \quad (-, -)$

III. $0 < y < 1 \quad \ln y < 0; x < 0 \quad (+, -)$

IV. $0 < y < 1 \quad \ln y < 0; x > 0 \quad (+, +)$