

Analysis III: Auditorium exercise class

Contour lines, gradients, higher-order derivatives

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BITTE BEACHTEN SIE DIE 3G-REGEL!

PLEASE OBEY THE 3G RULE!



Zutritt zur Lehrveranstaltung
haben nur:

- VOLLSTÄNDIG GEIMPFTE
- GENESENE
- GETESTETE

(negatives Testergebnis ist max. 24 Std. gültig)

Sollten Sie dies nicht nachweisen
können, müssen Sie bitte den Raum
jetzt verlassen.
Andernfalls droht ein Hausverbot!

Vielen Dank für Ihr Verständnis.
Schützen Sie sich und andere!

Admission to the course is restricted
to persons who are:

- FULLY VACCINATED
- RECOVERED
- TESTED

(negative test result is valid for max. 24 hours)

If you cannot prove this,
please leave the room now.
Otherwise you could be banned from
the room!

Thank you for your understanding.
Protect yourself and others!

Organisation

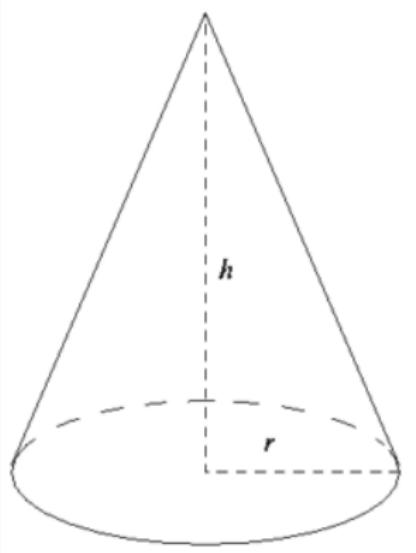
- office hours
 - Mon 13:15-14:15 bi-weekly (18.10., 01.11., 15.11. ...)
 - Location: SBC 3-E, Room 4.012
 - or in BBB
 - appointment by email sofiya.onyshkevych@uni-hamburg.de
- Room Change: Mon, 4pm - N0009. Last class - A0.18.

Setting

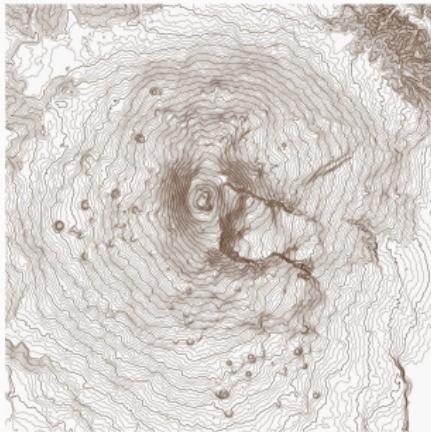
Given: $f : D \rightarrow \mathbb{R}, D \subset \mathbb{R}^n$

$$V(h, r) = \frac{1}{3}\pi r^2 h$$

$$D := \{(h, r) \in \mathbb{R}^2 : h > 0, r > 0\}$$



What is a level curve?



(a) A relief image of Mount Etna.
Source: Wikipedia



(b) A perspective photo of Etna.
Source: Wikipedia

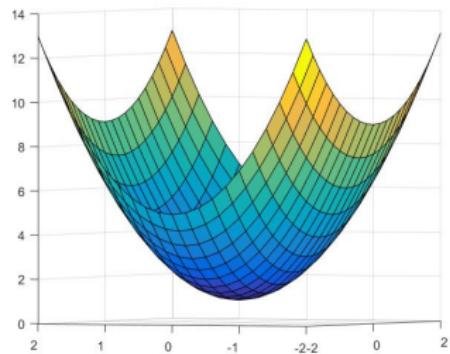
Example 1

Plot contour lines of the given function

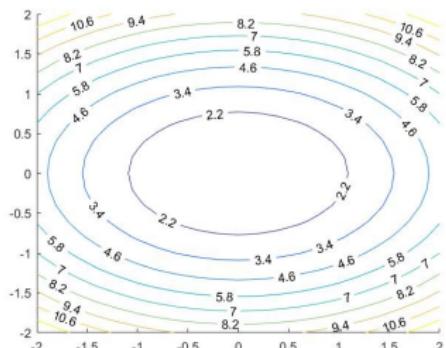
$$z = f(x, y) = x^2 + 2y^2 + 1.$$

Solution:

Example 1



(a) Surf plot of f .



(b) Contour plot of f .

Exercise 1

Plot contour lines of the given function

$$z = f(x, y) = x + y^2 + 1$$

Solution:

Exercise 2

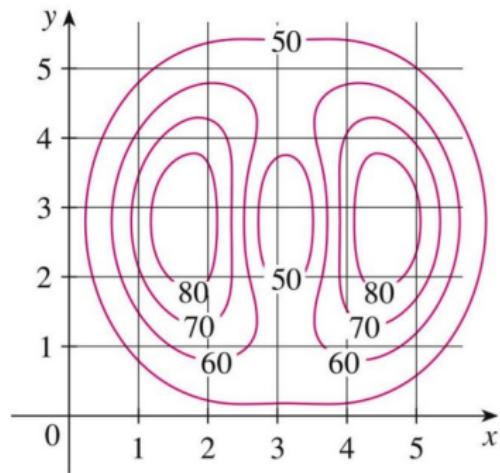
Plot contour lines of the given function and determine the direction of the gradient

$$z = f(x, y) = x + y^2 + 1$$

Solution:

Exercise 3

Given the contour map¹ for a function f , estimate the values of $f(1, 3)$ and $f(4, 5)$. What can you say about the shape of the graph of f ?



¹<https://www.usna.edu/Users/math/uhan/sm223/lessons/12%20Level%20Curves.pdf>

Plotting in Matlab

```
x=[ -2 : .2 : 2];  
y=[ -2 : .2 : 2];  
[X,Y] = meshgrid(x,y);  
Z=X.^2 + 2*Y.^2 + 1;  
mesh(X,Y,Z)  
contour(X,Y,Z,30)
```

Continuity and Differentiability

Let $D \subset \mathbb{R}^n$ —open, $f: D \rightarrow \mathbb{R}$, $x_0 \in D$.

f is called **partially differentiable in x_0 with respect to x_i** if the limit

$$\frac{\partial f}{\partial x_i}(x_0) := \lim_{t \rightarrow 0} \frac{f(x_0 + te_i) - f(x_0)}{t}$$

exists. Here e_i denotes the i —th unit vector. This limit is called **partial derivative** of f with respect to x_i at x_0 .

f is (continuously) **partially differentiable** if f is (continuously) partially differentiable w.r.t. each component x_1, \dots, x_n .

Examples

Are functions $f_1 - f_3$ partially differentiable? Compute the partial derivatives of the given functions.

- $f_1(x_1, x_2) = 3x_1^3 + 8x_2^2$
- $f_2(x_1, x_2) = x_1^4 - |x_2|$
- $f_3(x_1, x_2) = 3x_1 - 5x_2$

Gradient and nabla-operator

Let $D \subset \mathbb{R}^n$ – open, $f: D \rightarrow \mathbb{R}$ partially differentiable.

- We denote the row vector

$$\text{grad } f(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right)$$

as **gradient** of f at x_0 .

- We denote the symbolic vector

$$\nabla := \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n} \right)^T$$

as **nabla-operator**

- Thus we obtain **the column vector**

$$\nabla f(x_0) := \left(\frac{\partial f}{\partial x_1}(x_0), \frac{\partial f}{\partial x_2}(x_0), \dots, \frac{\partial f}{\partial x_n}(x_0) \right)^T$$

Examples

Compute the gradient of the given functions

More Examples

Compute the gradient of the given function

$$f(x, y, z) = x^2 y \cos(z)$$

$$f_x(x, y, z) =$$

$$f_y(x, y, z) =$$

$$f_z(x, y, z) =$$

$$\text{grad } f(x) = (f_{x_1}(x), \dots, f_{x_n}(x)) =$$

Tangent Planes

Let $P_0 = (x_0, y_0, z_0)$ be a point on a surface S , and let C be any curve passing through P_0 and lying entirely in S . If the tangent lines to all such curves C at P_0 lie in the same plane, then this plane is called the **tangent plane** to S at P_0 .

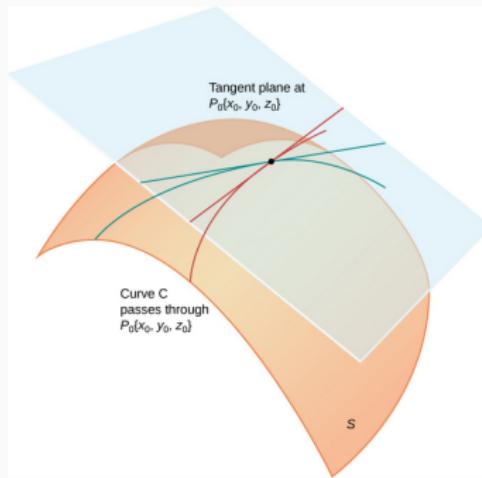


Figure 3: The tangent plane to a surface S at a point P_0 ²

²<https://math.libretexts.org>

Exercise

Compute the tangent plane to the graph f at point $(x_0, y_0) := (1, 3)$.

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y) = 4 - x^2 - y^2$$

Solution:

$$T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f(1, 3) =$$

$$f_x(x, y) =$$

$$f_y(x, y) =$$

Exercise

Compute the tangent plane to the surface $z = x^2 + y^2$ at the point $(x_0, y_0, z_0) := (1, 2, 5)$.

Solution:

$$T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Exercise 2 (part 2)

Plot contour lines of the given function and determine the direction of the gradient

$$z = f(x, y) = x + y^2 + 1$$

Solution:

Higher-order derivatives

Compute the derivatives of the second order of the given function

$$f(x, y, z) = x^2 y \cos(z)$$

$$f_x(x, y, z) = y^2 \cos(z)$$

$$f_y(x, y, z) = 2xy \cos(z)$$

$$f_z(x, y, z) = xy^2 (-\sin(z))$$

$$\frac{\partial}{\partial x} f_y(x, y, z) =$$

$$\frac{\partial}{\partial y} f_y(x, y, z) =$$

$$\frac{\partial}{\partial z} f_y(x, y, z) =$$

Hessian Matrix

If all second partial derivatives of f exist and are continuous over the domain of the function, then **the Hessian matrix H** of f is a square $n \times n$ matrix:

$$\text{Hess } f(\mathbf{x}) = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n \partial x_n} \end{pmatrix} (\mathbf{x})$$

Exercise

Compute first and second derivatives of f

$$f: \mathbb{R}^2 \rightarrow \mathbb{R}, \quad f(x, y, z) = z^2 + \ln(xy^2)$$

Exercise

Compute the third order partial derivative f_{xyy} of a function f .

$$f(x, y) = \ln(xy^2 + 2y)$$

Exercise

Compute the first and second order partial derivatives of a function f :

$$f(x, y) = 8x - 2x^2y^2$$

Exercise

Compute the first and second order partial derivatives of a function f for $x \neq 0$:

$$f(x, y, z) = e^z + \frac{1}{x} + xe^{-y}$$

Some more exercises

Compute partial derivatives of given functions

- $(3x - 5y)^4$
- $\frac{\sin(xyz)}{x^2}$
- $\frac{y}{x} e^{-(x^2+y^2)}$
- $\arctan \frac{x}{y}$

Exercise

Compute the tangent plane to the surface defined by the function $f(x, y) = \sin(2x) \cos(3y)$ at the point $(\pi/3, \pi/4)$. Solution:

$$T(x, y) := f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Thank you!