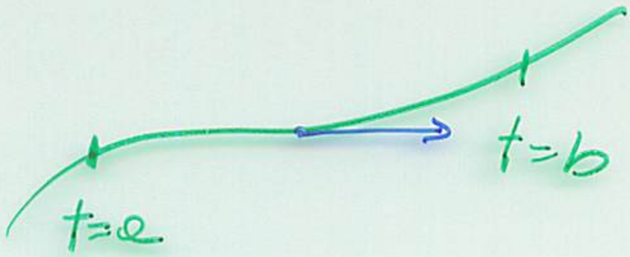
10.06 ①

$C: [a,b] \to \mathbb{R}^N$

$c = c(t)$

$\dot{c} = \dot{c}(t)$

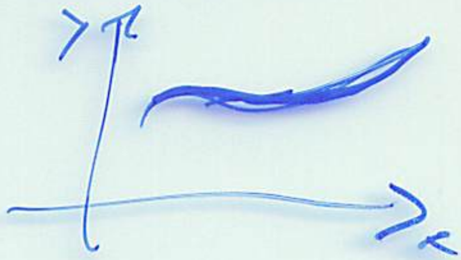


Bspl

$f = f(x)$ $\qquad f: [a,b] \to \mathbb{R}$

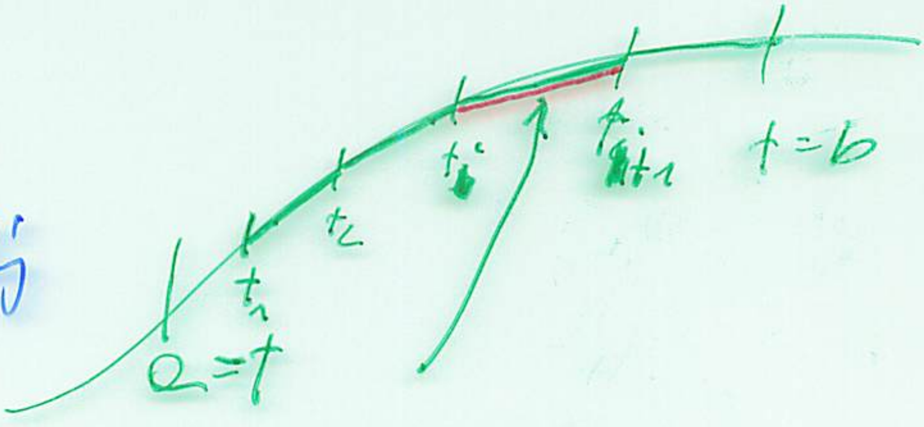
$c(x) = (x, f(x))$

$C: [a,b] \to \mathbb{R}^2$



Länge einer Kurve

Unterteilung



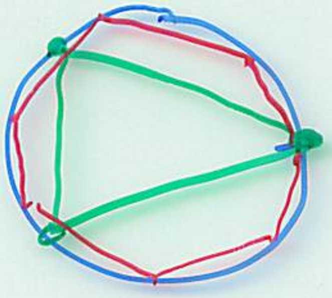
$\| c(t_{i+1}) - c(t_i) \|$

$$L(Z) = \sum_{i=0}^{n-1} \| c(t_{i+1}) - c(t_i) \|$$

$\forall Z \qquad L(Z) \le$ ~~Kurvenlänge~~ Kurvenlänge $\qquad$ 10. VO ②

Bspl

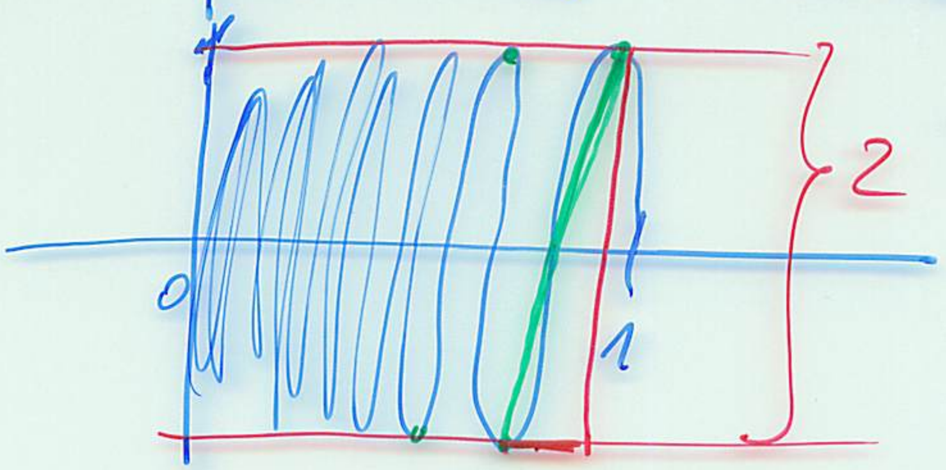
$$c(t) = r\begin{pmatrix}\cos t\\ \sin t\end{pmatrix} \qquad t \in [0, 2\pi]$$



$$\sup_Z L(Z) = 2r\pi$$

Bspl:

$$f(x) = \sin\frac{1}{x} \qquad x \in (0, 1]$$



$$c(x) = (x, f(x))$$

Z:

$$\frac{1}{x_n} = (2n-1)\frac{\pi}{2}, \qquad x_n = \frac{1}{(2n-1)\frac{\pi}{2}}$$

$$L(Z) = \sum_{n=1}^{\infty} \underbrace{\| c(x_{n+1}) - c(x_n) \|}_{\ge 2} = \sum_{n=1}^{\infty} 2 \text{ div}$$

$$L(Z) = \sum_{n=\ldots}^{\infty} \underbrace{\left\| \frac{c(t_{i+1}) - c(t_i)}{\Delta t_i} \right\|}_{\approx \dot c(t_i)} \underbrace{\Delta t_i}_{dt} \to \int_a^b \|\dot c(t)\| \, dt$$

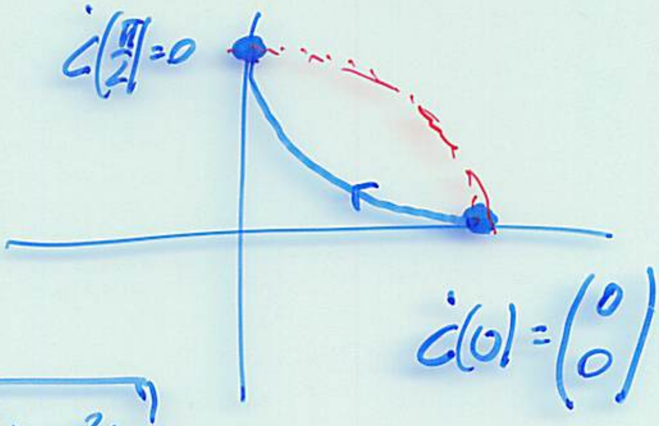
10Vo ③

Astroide

$x = \cos^3 t$

$y = \sin^3 t$

$t \in [0, \frac{\pi}{2}]$



$$c(t) = \begin{pmatrix} \cos^3 t \\ \sin^3 t \end{pmatrix}$$

$$\dot{c}(t) = \begin{pmatrix} -3\cos^2 t \sin t \\ 3\sin^2 t \cos t \end{pmatrix}$$

$$\|\dot{c}(t)\| = 3\sqrt{\cos^4 t \sin^2 t + \sin^4 t \cos^2 t} =$$

$$= 3\cos t \sin t$$

$$L = \int_0^{\pi/2} 3\cos t \sin t \, dt = \frac{3}{2} \sin^2 t \Big|_0^{\frac{\pi}{2}} = \frac{3}{2}$$

$t = t(\tau)$ $\qquad t : [\alpha, \beta] \to [a, b]$ $\quad C^1$

$\tilde{c}(\tau) = c(t(\tau))$ $\qquad (\;)^{\cdot} = \frac{d}{dt}$

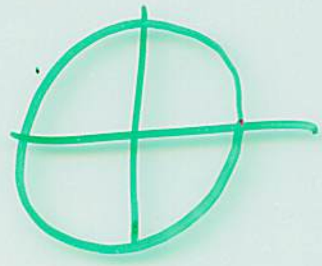
$\tilde{c}'(\tau) = \dot{c}(t(\tau)) \cdot t'(\tau)$ $\qquad (\;)' = \frac{d}{d\tau}$

$$L = \int_\alpha^\beta \|\tilde{c}'(\tau)\| d\tau =$$

$$= \int_\alpha^\beta \|\dot{c}(t(\tau))\, \underbrace{t'(\tau)}\| d\tau = \int_\alpha^\beta \|\dot{c}(t(\tau))\|\, t'(\tau)\, d\tau$$

$$= \left[\int_a^b \|\dot{c}(t)\| dt = L \right.$$

Bsp



$$c(\varphi) = \begin{pmatrix} R\cos\varphi \\ R\sin\varphi \end{pmatrix}$$

$$\varphi \in [0, 2\pi]$$

Länge $2R\pi$

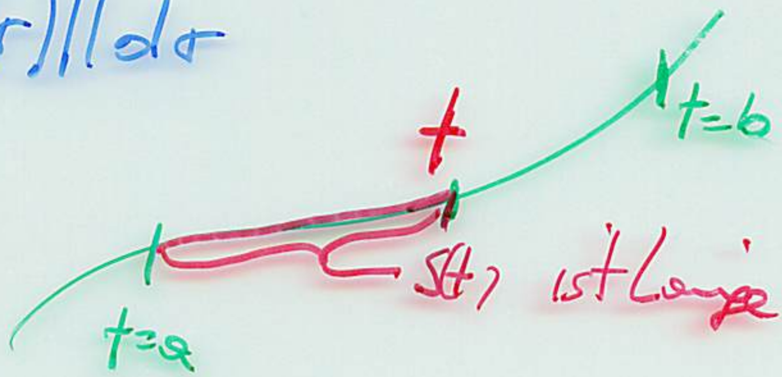
φ durchläuft 2π

---

c

$$s(t) = \int_a^t \|\dot c(\tau)\|\, d\tau$$

$$\dot s(t) = \|\dot c(t)\|$$



s monoton steigend

c C^1 Kurve

$s^{-1}(s)$

$( )' = \frac{d}{ds}$

$\dot{( )} = \frac{d}{dt}$

$$\tilde c(s) = c(s^{-1}(s))$$

$$\tilde c'(s) = \dot c(s^{-1}(s)) \underbrace{(s^{-1}(s))'}_{\frac{1}{\dot s(t)}} = \boxed{\frac{\dot c(s^{-1}(s))}{\|\dot c(s^{-1}(s))\|}}$$

Einheitsvektor

Astroide 10.V. 5

$$c(t) = \begin{pmatrix} \cos^3 t \\ \sin^3 t \end{pmatrix}$$

$$s = S(t) = \int_0^t \underbrace{\|\dot{c}(\sigma)\| d\sigma}_{3 \sin t \cos t} = \frac{3}{2} \sin^2 t$$

$$t = S^{-1}(s) = \arcsin\left(\sqrt{\tfrac{2}{3} s}\right)$$

$$\tilde{c}(s) = c(S^{-1}(s)) = \begin{pmatrix} \cos^3\left(\arcsin\sqrt{\frac{2}{3}s}\right) \\ \sin^3\left(\arcsin\sqrt{\frac{2}{3}s}\right) \end{pmatrix}$$

$$= \begin{pmatrix} \left(1 - \frac{2}{3}s\right)^{\frac{3}{2}} \\ \left(\sqrt{\frac{2}{3}s}\right)^3 \end{pmatrix} = \begin{pmatrix} \left(1 - \frac{2}{3}s\right)^{\frac{3}{2}} \\ \left(\frac{2}{3}s\right)^{3/2} \end{pmatrix}$$

$$\kappa(s) = \|\tilde{c}''(s)\| \qquad (\ )' = \frac{d}{ds}$$

$$c = c(x) = (x, f(x)) \qquad (\ )^{\cdot} = \frac{d}{dx}$$

$$\dot{c} = \dot{c}(x) = (1, f')$$

$$\ddot{c} = \quad = (0, f'')$$

10Vo 6

$$S(x) = \int_0^x \|\dot{c}(y)\| dy$$

$$\dot{S}(x) = \|\dot{c}(x)\| = \sqrt{1+f'^2}$$

$$\tilde{c}(s) = c(S^{-1}(s))$$

$$\tilde{c}'(s) = \frac{\dot{c}(S^{-1}(s))}{\|\dot{c}(S^{-1}(s))\|}$$

$$\tilde{c}''(s) = \frac{\ddot{c}}{\| \ \|^2} - \frac{\dot{c}}{\| \ \|^2} \frac{\cancel{2}\langle \dot{c}\cdot\ddot{c}\rangle}{\cancel{2}\| \ \|} \cdot \underbrace{\frac{1}{\| \ \|}}_{(S^{-1}(s))' = \frac{1}{\dot{S}(x)}} = (*)$$

$$\|\dot{c}\| = \sqrt{\langle \dot{c}, \dot{c}\rangle}$$

$$\langle \dot{c}\cdot\ddot{c}\rangle = f'f''$$

$$(*) = \frac{\ddot{c}}{\| \ \|^2} - \frac{\dot{c}\, f'f''}{\| \ \|^4} = \left(\frac{1}{\| \ \|^2}(0, f'') - \frac{(1, f')\, f'f''}{\| \ \|^4}\right)$$

$$\kappa(s) = \|\tilde{c}''(s)\| = \sqrt{\frac{(f'f'')^2}{\| \ \|^8} + \frac{1}{\| \ \|^8}\left(f''\underbrace{\| \ \|^2}_{1+f'^2} - f'f'f''\right)^2}$$

$$= \frac{1}{\| \ \|^4}\sqrt{f'^2 f''^2 + f''^2} =$$

$$= \frac{1}{\| \ \|^4}\sqrt{1+f'^2}\cdot|f''|$$

$$= \boxed{\frac{|f''|}{\left(\sqrt{1+f'^2}\right)^3}}$$

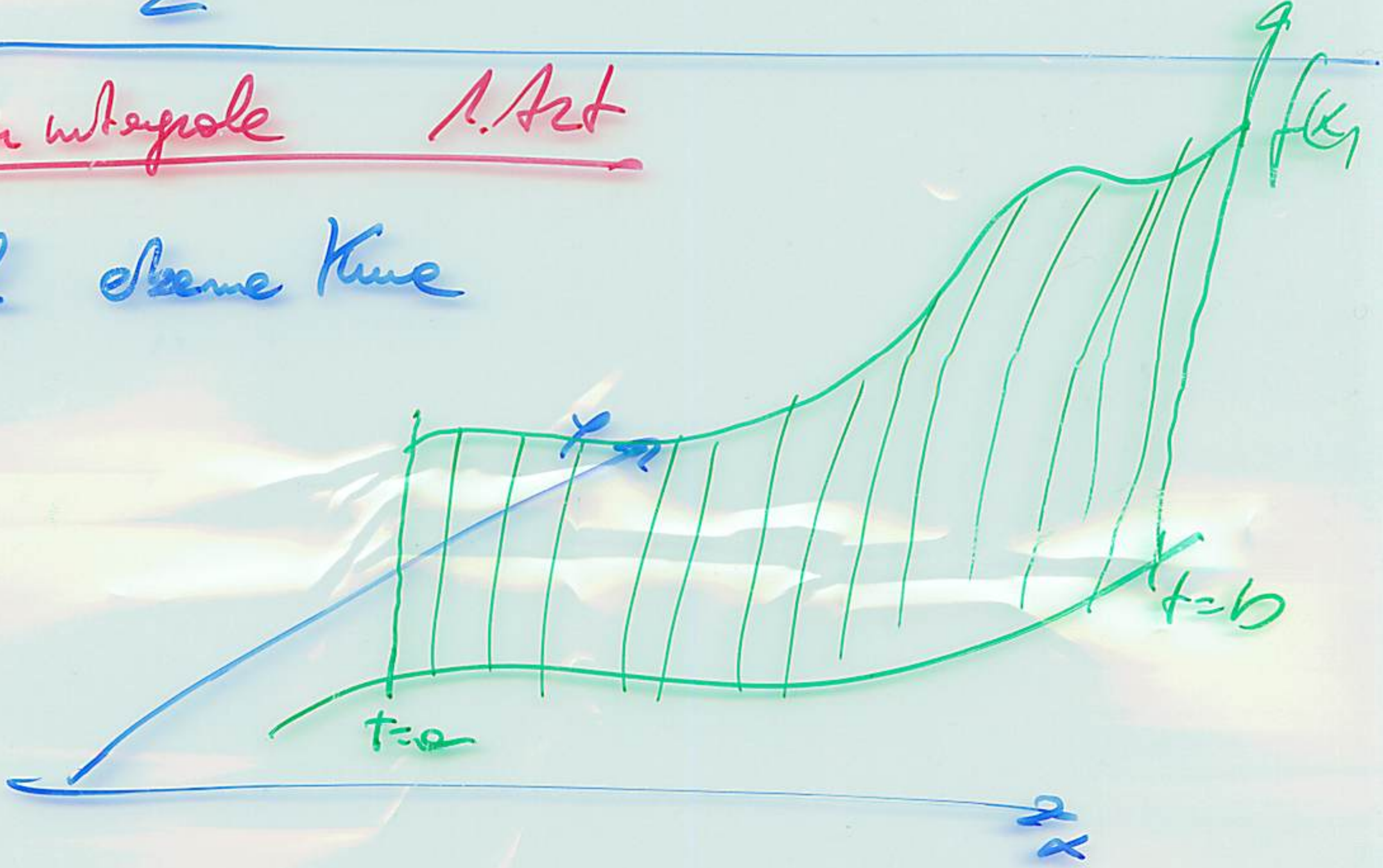
$$c(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = r \begin{pmatrix} \cos t \\ \sin t \end{pmatrix}$$

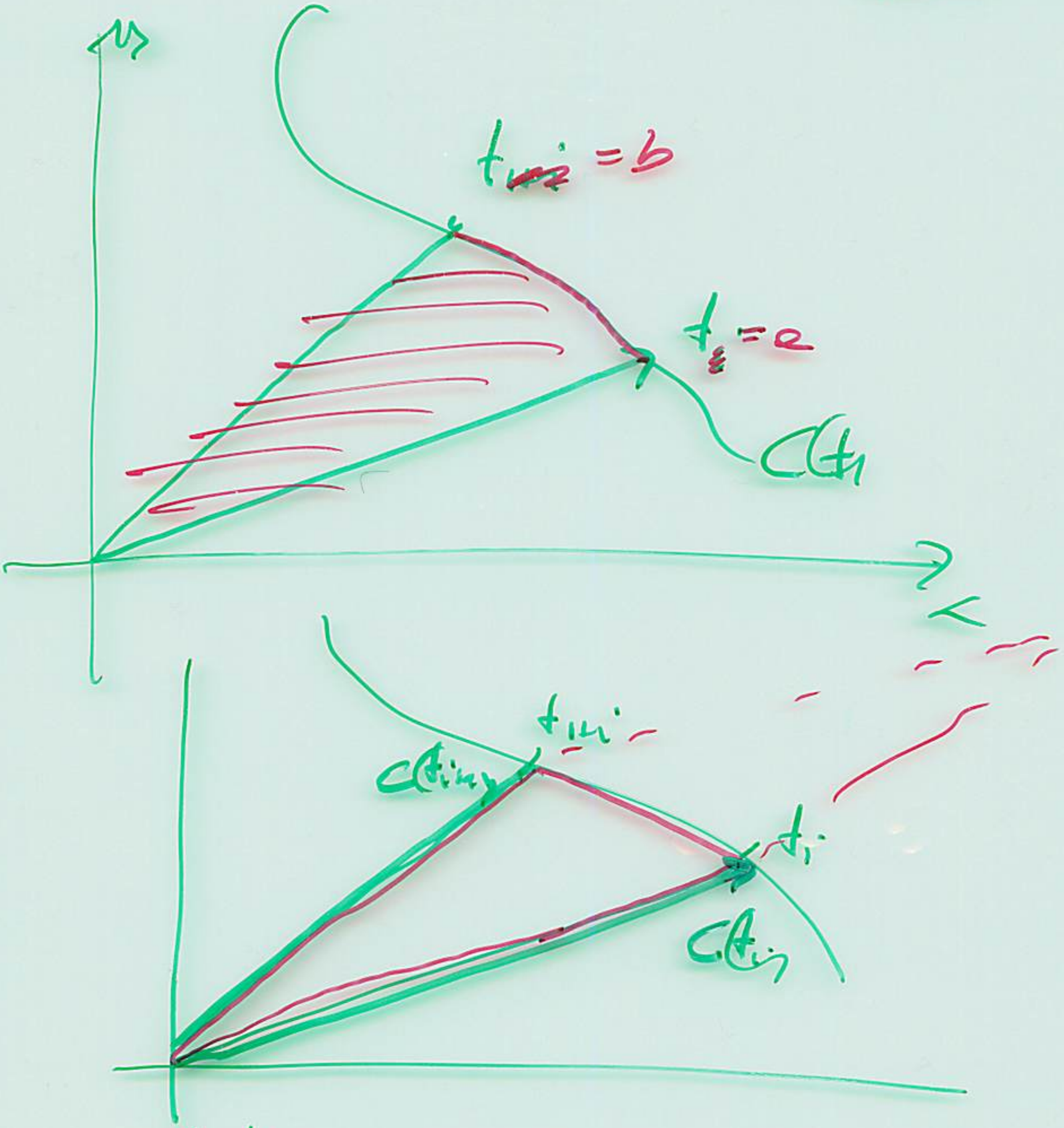
$$F = \frac{1}{2} \int_0^{2\pi} (r\cos t \, r\cos t + r\sin t \, r\sin t)\, dt =$$

$$= \frac{r^2}{2} \cdot 2\pi = r^2 \pi$$

Kurvenintegrale 1. Art

Bsp ebene Kurve





$$Fl_{\Delta} = \frac{1}{2} \left\| C(t_i) \times C(t_{i+1}) \right\|$$

$$= \frac{1}{2} \left\| \begin{pmatrix} x(t_i) \\ y(t_i) \end{pmatrix} \times \begin{pmatrix} x(t_{i+1}) \\ y(t_{i+1}) \end{pmatrix} \right\|$$