Konvergenzradius

$$r = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right| = \lim_{k\to\infty} \frac{(k+1)!}{k!} = \lim_{k\to\infty} (k+1) = \infty$$

Ableitung:

$$\frac{d}{dx} \exp(x) = \frac{d}{dx} \left(\sum_{k=0}^{\infty} \frac{1}{k!} x^k \right) = \sum_{k=1}^{\infty} \underbrace{\frac{1}{k!} k}_{\frac{1}{(k-1)!}} \cdot x^{k-1}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} x^k = \exp(x)$$

Lösung der DGL:

Lösungsansatz $y(x) = c e^{\lambda x}$ $\quad c, \lambda$ zu bestimmen

DGl einsetzen $\underbrace{\lambda c e^{\lambda x}}_{y'} = a \underbrace{c e^{\lambda x}}_{y} \Rightarrow \lambda = a$

Anfangsbed.

$$y_0 = y(x_0) = c e^{a x_0} \Rightarrow c = y_0 e^{-a x_0}$$

$$\Rightarrow y(x) = y_0 e^{-a x_0} e^{a x} = y_0 e^{a(x - x_0)}$$

1. Funktionalgleichung $z, w \in \mathbb{C}$

$$\exp(z)\exp(w) = \sum_{n=0}^{\infty} \frac{1}{n!} z^n \cdot \sum_{n=0}^{\infty} \frac{1}{n!} w^n$$

$\downarrow$ Cauchy Prod.

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{1}{(n-k)!} z^{n-k} \frac{1}{k!} w^k$$

$\downarrow$ Binom Koeff.

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} z^{n-k} w^k$$

$\downarrow$ Binom. Lehrsatz

$$= \sum_{n=0}^{\infty} \frac{1}{n!} (z+w)^n = \exp(z+w)$$

~~$\forall z \in \mathbb{C}$~~

2) $1 = \exp(0) = \exp(z-z) = \exp(z)\exp(-z), \forall z \in \mathbb{C}$

$\Rightarrow \exp(z) \neq 0 \quad \Rightarrow \quad \exp(-z) = \dfrac{1}{\exp(z)}$

3) $x \geq 0: \quad \exp(x) = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots \geq 1 > 0$

$\Rightarrow \exp(-x) = \dfrac{1}{\exp(x)} > 0 \Rightarrow \forall x \in \mathbb{R} \ \exp(x) > 0$

4) $\exp(x) = 1 + x + \dfrac{x^2}{2!} + \cdots \overset{x \geq 0}{\geq} 1 + x \xrightarrow{x \to \infty} \infty$

5) $n \in \mathbb{N}$

$$\lim_{n \to \infty} \frac{x^n}{\exp(x)} = 0$$ L'Hospital

6) $\exp'(x) = \exp(x) > 0$ streng monoton wachsend $\forall x \in \mathbb{R}$

8) $q = \frac{n}{m} \in \mathbb{Q}$; $x \in \mathbb{R}$

$$\exp(q \cdot x) = \exp\left(\tfrac{n}{m} x\right)$$

Funktionalgl.

$$= \exp\Big(\underbrace{\tfrac{1}{m}x \cdot \ldots \cdot \tfrac{1}{m}x}_{n\text{-mal}}\Big) = \exp\left(\tfrac{x}{m} \cdots \tfrac{x}{m}\right)^n$$

$$= \exp(x \cdots x)^{\frac{n}{m}}$$

$$\overset{\exp(x)}{\overset{\shortparallel}{\exp\left(m \tfrac{x}{m}\right)}} = \exp\left(\tfrac{x}{m}\right)^m \Rightarrow \exp\left(\tfrac{x}{m}\right) = \left(\exp(x)\right)^{\frac{1}{m}}$$

Umkehrfunktion zu exp(x)

$$\exp : \mathbb{R} \to \,]0, \infty[$$

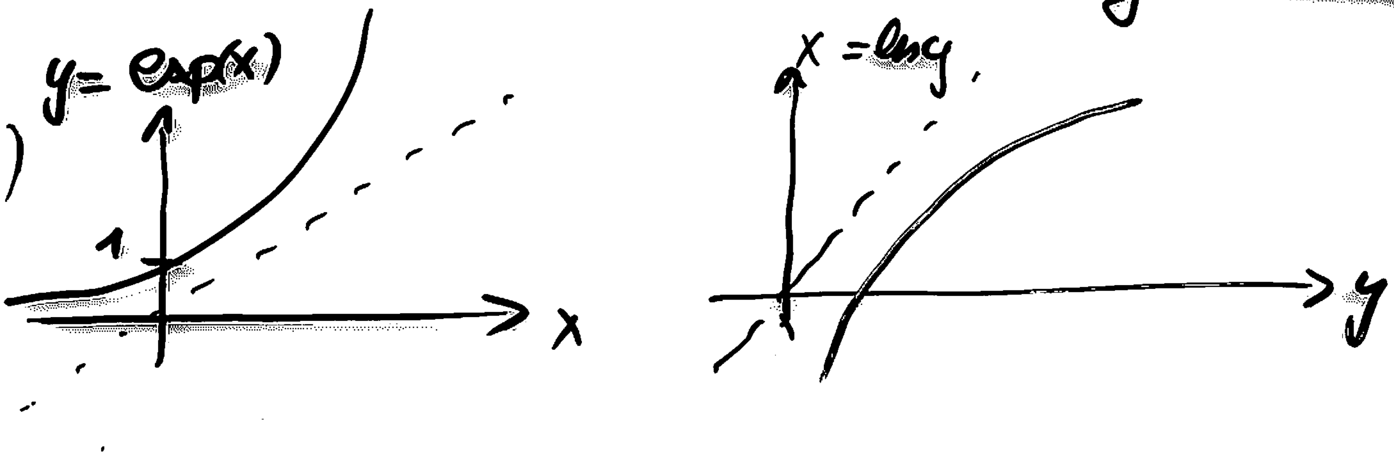
$$\mathbb{R} \leftarrow \,]0, \infty[\; : \ln$$

1) + 6) $$(\ln x)' = \frac{1}{\exp' y \,|_{y = \ln x}} = \frac{1}{\exp(\ln x)} = \frac{1}{x}$$

> 0

Streng m. wachsend

2)



3) $x_1, x_2 \in \mathbb{R}$ mit $y_1 = e^{x_1} \Leftrightarrow x_1 = \ln y_1$

$y_2 = e^{x_2} \Leftrightarrow x_2 = \ln y_2$

$$\Rightarrow \exp(x_1 + x_2) = \exp(x_1)\exp(x_2)$$

$$= \exp(\ln y_1) \cdot \exp(\ln y_2) = y_1 \cdot y_2$$

$$\Rightarrow x_1 + x_2 = \ln y_1 \cdot y_2 = \ln y_1 + \ln y_2$$

7) $\ln(1+x) = \int (\ln(1+x))' dx = \int \frac{1}{1+x} dx$

geom. Reihe $\overset{|x|<1}{=} \int \sum_{k=0}^{\infty} (-x)^k dx$

gliedweise integrieren $\overset{\curvearrowright}{=} \sum_{k=0}^{\infty} \int (-x)^k dx$

integrieren $= \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1} + C$

zur Konstanten

$\ln(1+0) = 0 = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} 0^{k+1}}_{=0} + C$

$\Rightarrow C = 0$

1)+4) $f'(x) = (a^x)' = \exp(x \ln a)'$

$= \ln a \exp(x \ln a)$

$= \underbrace{(\ln a)}_{} \cdot a^x \begin{cases} > 0 & a > 1 \\ < 0 & 0 < a < 1 \end{cases}$

3) $y(x) = \log_a x \Rightarrow a^{y(x)} = x \Rightarrow y(x) \ln a = \ln x$ 41

$$\Rightarrow y(x) = \frac{\ln x}{\ln a}$$

4) $$\frac{d}{dx}\left(x^a\right) = \left(e^{\ln x^a}\right)' = \left(e^{a \ln x}\right)'$$

$$= a \cdot \frac{1}{x} e^{a \ln x} = a \frac{1}{x} x^a$$

$$= a x^{a-1}$$

$$g(x) = \sum_{k=0}^{\infty} \binom{a}{k} x^k$$

42 a)

$$a_k = \binom{a}{k} := \frac{1}{k!} \prod_{j=0}^{k-1} (a-j)$$

$$a_0 = \binom{a}{0} = \underbrace{\underbrace{\frac{1}{0!}}_{=1} \underbrace{\prod_{j=0}^{-1} (a-j)}_{\text{leeres Produkt}}}_{=1}$$

Konvergenzradius

$$r = \lim_{k\to\infty} \left| \frac{a_k}{a_{k+1}} \right|$$

$$= \lim_{k\to\infty} \left| \frac{\prod_{j=0}^{k-1}(a-j) \; (k+1)!}{k! \prod_{j=0}^{k}(a-j)} \right|$$

$$= \lim_{k\to\infty} \left| \frac{k+1}{a-k} \right| = 1$$

$\Rightarrow$ gliedweises differenzieren erlaubt : $|x| < 1$

$$g'(x) = \left(\sum_{k=0}^{\infty} \binom{a}{k} x^k \right)' = \sum_{k=1}^{\infty} \binom{a}{k} k x^{k-1}$$

$$\Rightarrow (1+x) g'(x) = (1+x) \sum_{k=1}^{\infty} \binom{a}{k} k x^{k-1}$$

$$= \sum_{k=1}^{\infty} \binom{a}{k} k x^{k-1} + \sum_{k=1}^{\infty} \binom{a}{k} k x^{k}$$

42 b

$$= \sum_{k=0}^{\infty} \binom{a}{k+1}(k+1)x^k + \sum_{k=1}^{\infty} \binom{a}{k} k x^k$$

$$= a + \sum_{k=1}^{\infty} \underbrace{\left\{ \binom{a}{k+1}(k+1) + \binom{a}{k} k \right\}}_{\stackrel{\cdots}{=} \frac{a}{k!} \prod_{j=0}^{k-1} (a-j)} x^k$$

$$= a + \sum_{k=1}^{\infty} \frac{a}{k!} \prod_{j=0}^{k-1} (a-j) \; x^k$$

$$= a\left(1 + \sum_{k=1}^{\infty} \binom{a}{b} x^k\right) = a\, g(x)$$

Insgesamt

$$\Rightarrow (1+x) g'(x) = a\, g(x) \quad \text{"DGL"}$$

Lösen der DGL durch Trennung der Veränderlichen

$$\Rightarrow \frac{g'(x)}{g(x)} = \frac{a}{1+x} \quad \Big| \int dx$$

$$\Rightarrow \int \frac{g'(x)}{g(x)} \underbrace{g'(x)\,dx}_{=dg \text{ Subst.}} \stackrel{\text{Regel}}{=} \int \frac{dg}{g} = \int \frac{a}{1+x} dx$$

$$\Rightarrow \ln|g(x)| = a \cdot \ln(1+x) + \tilde{C} = \ln(1+x)^a + \tilde{C}$$

$$\Rightarrow |g(x)| = e^{\ln(1+x)^a + \tilde{C}} = e^{\ln(1+x)^a} \cdot \underbrace{e^{\tilde{C}}}_{= C > 0}$$

$$\Rightarrow g(x) = \pm C \, (1+x)^a$$

<u>zur Konstanten:</u>

$$\Rightarrow \underbrace{g(0)}_{\binom{a}{0} = 1} = \pm C \, (1+0)^a = \pm C \Rightarrow C = 1$$

$$\Rightarrow (1+x)^a = g(x) = \sum_{k=0}^{\infty} \binom{a}{k} x^k$$

42 d)

$$\sqrt{1+x} = (1+x)^{1/2} = \sum_{k=0}^{\infty} \binom{1/2}{k} x^k$$

mit $\binom{1/2}{0} = 1$; $\binom{1/2}{1} = \frac{1/2}{1} = \frac{1}{2}$

$$\binom{1/2}{2} = \frac{1/2 \cdot (-1/2)}{2} = -\frac{1}{8} \quad \ldots$$

$$\cosh z := \frac{1}{2}\left(e^{z}+e^{-z}\right)$$

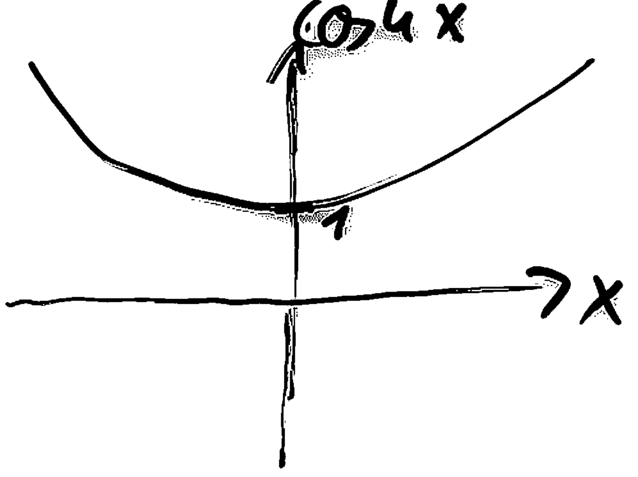
$$= \frac{1}{2}\left(\sum_{k=0}^{\infty}\frac{z^k}{k!}+\sum_{k=0}^{\infty}\frac{(-z)^k}{k!}\right)$$

$$= \frac{1}{2}\sum_{k=0}^{\infty}\underbrace{\frac{\left(1+(-1)^k\right)}{k!}}_{\begin{cases}2 & k \text{ gerade: } k=2j\\ 0 & k \text{ ungerade: } k=2j+1\end{cases}} z^k$$

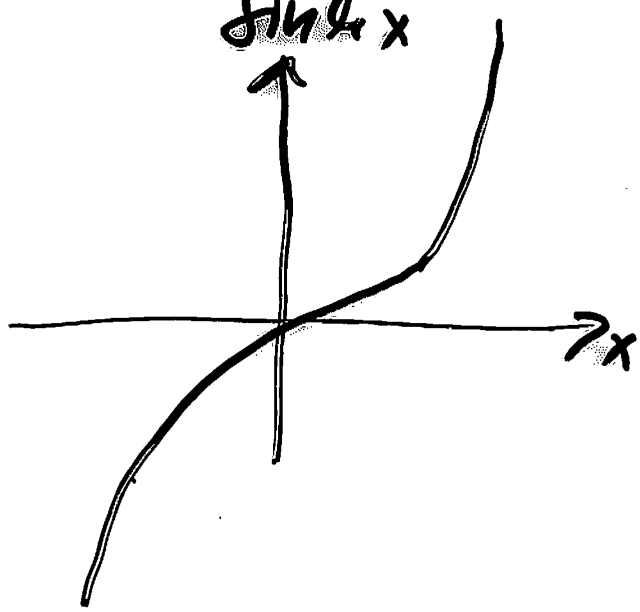
$$= \cancel{\frac{1}{2}}\sum_{j=0}^{\infty}\frac{\cancel{2}}{(2j)!}\,x^{2j} = \sum_{j=0}^{\infty}\frac{x^{2j}}{(2j)!}$$

↑ nur gerade Potenzen

→ cosh z ist gerade



$x \in \mathbb{R}$



2) $(\cosh x)' = \frac{1}{2}(e^{x} + e^{-x})'$

$= \frac{1}{2}(e^{x} - e^{-x}) = \sinh x$

$(\sinh x)' = \left(\frac{1}{2}(e^{x} - e^{-x})\right)' = \frac{1}{2}(e^{x} + e^{-x})$

$= \cosh x$

$$y = \operatorname{arsinh} x \Leftrightarrow \sinh y = x$$

$$\Rightarrow \quad x^2 + 1 = \sinh^2 y + 1 = \cosh^2 y$$

$$\Rightarrow \quad \cosh y = +\sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$$

da $\cosh y > 0$

$$\Rightarrow \quad x + \sqrt{1+x^2} = \sinh y + \cosh y$$

$$= \frac{1}{2}\left(e^{y} - \cancel{e^{-y}}\right) + \frac{1}{2}\left(e^{y} + \cancel{e^{-y}}\right)$$

$$= e^{y}$$

$$\Rightarrow \operatorname{arsinh} x = y = \ln\left(x + \sqrt{1+x^2}\right)$$

2) $e^{iz} = \sum_{k=0}^{\infty} \frac{1}{k!}(iz)^k$

$= \sum_{k=0}^{\infty} \frac{(i)^k}{k!} z^k \qquad (i^k) = \begin{cases} (i)^{2j} = (-1)^j \,; & k=2j \\ (i)^{2j+1} = i(-1)^j & k=2j+1 \end{cases}$

$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j} + \frac{i(-1)^j}{(2j+1)!} z^{2j+1}$

$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} z^{2j} + i \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!} z^{2j+1}$

$= \cos z + i \sin z$

$\cot z = \frac{\cos z}{\sin z}$ Definitionslücken in $\mathbb{C}$

für $0 = \sin z = \frac{1}{2i}\left(e^{iz} - e^{-iz}\right)$

$z = x + iy$

$y, x \in \mathbb{R}$

$$= \frac{1}{2i}\left(e^{i(x+iy)} - e^{-i(x+iy)}\right)$$

$$= \frac{1}{2i}\left(e^{ix}e^{-y} - e^{-ix}e^{y}\right)$$

$$= \frac{1}{2i}\left((\cos x + i\sin x)e^{-y} - (\cos x - i\sin x)e^{y}\right)$$

$$= \sin x\left(\frac{1}{2}\left(e^{y} + e^{-y}\right)\right) + \frac{1}{2i}\cos x\left(e^{-y} - e^{y}\right)$$

$$= \underbrace{\sin x\,\overbrace{\cosh y}^{\neq 0}}_{=0} + i\underbrace{\cos x \sinh y}_{=0}$$

$\Downarrow$ mit 1)

$\overset{1)}{\Rightarrow} \sin x = 0 \Rightarrow x = k \cdot \pi$

$\underbrace{\cos k\pi}_{(-1)^k} \cdot \underbrace{\sinh y}_{=0}$

$\Rightarrow y = 0$

$\Rightarrow z_k = k\pi + i \cdot 0 = \pi k$

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x}$$

$$= \begin{cases} 1 + \tan^2 x \\ \dfrac{1}{\cos^2 x} \end{cases}$$