Nachweis $\Gamma(x+1) = x\,\Gamma(x)$, $x > 0$,

wobei
$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$$

Gamma Funktion

$$\Gamma(x+1) = \int_0^\infty t^x e^{-t}\,dt$$

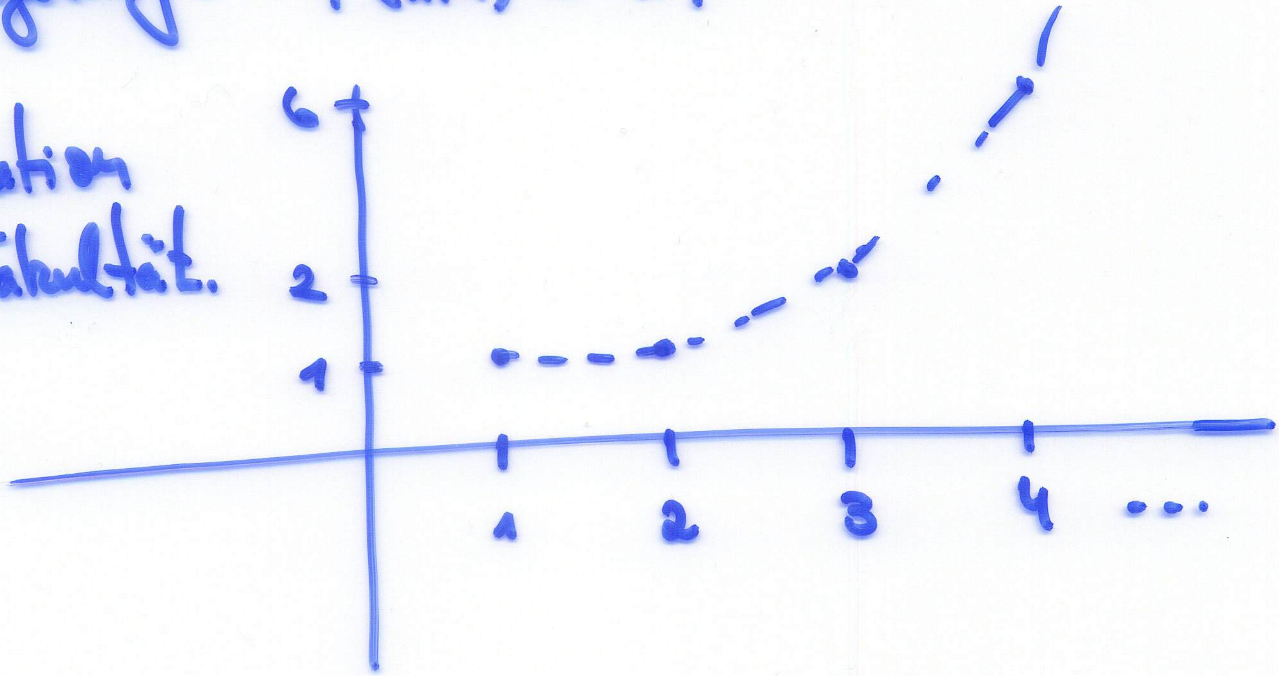
$$= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}} \int_\varepsilon^R t^x e^{-t}\,dt$$

$$\underbrace{\int_\varepsilon^R t^x e^{-t}\,dt}_{\xrightarrow[\varepsilon\to 0]{R\to\infty}\ \Gamma(x+1)} = \overset{*}{-t^x e^{-t}\Big|_\varepsilon^R} + x \underbrace{\int_\varepsilon^R t^{x-1} e^{-t}\,dt}_{\xrightarrow[\varepsilon\to 0]{R\to\infty}\ \Gamma(x)}$$

$$* = \underbrace{-R^x e^{-R}}_{\to 0\ (R\to\infty)} + \underbrace{\varepsilon^x e^{-\varepsilon}}_{\to 0\ (\varepsilon\to 0)}$$

Folgerung: $\Gamma(n+1) = n!$

Interpolation der Fakultät.




$\Gamma$ - Funktion ist ein Bsp für Parameter - abhängige Integrale

Beispiele

i.) $J_n(x) := \frac{1}{\pi}\int_0^\pi \cos(x \sin t - nt)\,dt$, $n \in \mathbb{N}$

Parameter hier: $x$

$J_n$ heißt $n$-te Bessel Funktion

ii) Laplace - Transformierte einer Funktion $f$

$$F(x) = \mathcal{L}(f)(x)$$

$$:= \int_0^\infty \cancel{f(x)}\, f(t)\, e^{-xt}\, dt$$

Parameter: $x$

iii) $\int \sin x \, t^2 \, dt = \overset{\text{Parameter: } x}{\sin x} \int t^2 \, dt$

$= \sin x \, \frac{1}{3} t^3 + C_1 = G(x)$

Fragen: $G(x)$ stetig ? ✓

$G(x)$ diffbar ?

$G'(x) = \frac{1}{3} \cos x \, t^3$ ✓

Allgemein: $\int_a^b f(x,t)\, dt =: G(x)$

$G$ stetig (bzgl. $x$), bzw diffbar

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$G$ stetig, falls $f$ stetig bzgl $x$ <u>und</u> $f(x,t)$ Riemann-integrierbar bzgl $t$.

$$G'(x) = \lim_{h\to 0} \frac{G(x+h) - G(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h} \left\{ \int_a^b f(x+h,t)\,dt - \int_a^b f(x,t)\,dt \right\}$$

$$= \int_a^b \lim_{h\to 0} \frac{f(x+h,t) - f(x,t)}{h}\,dt$$

$$= \int_a^b \frac{d}{dx} f(x,t)\,dt$$

<u>Merke</u>: $$\frac{d}{dx} \int_a^b f(x,t)\,dt = \int_a^b \frac{d}{dx} f(x,t)\,dt$$

Bsp: $G(x) = \int_a^b \sin x \, t^2 \, dt$

$f(x,t) = \sin x \, t^2$

$\frac{d}{dx} f(x,t) = \cos x \, t^2$

$G(x) = \frac{1}{3} t^3 \sin x \Big|_{t=a}^{t=b}$

$G'(x) = \frac{1}{3} t^3 \cos x \Big|_{t=a}^{t=b}$

$$\int_a^b \cos x \, t^2 \, dt = \int_a^b \frac{d}{dx} (\sin x \, t^2) \, dt$$

$$= \cos x \, \frac{1}{3} t^3 \Big|_{t=a}^{t=b} = G'(x).$$

Integrationsgrenzen können von x abhängen:

$$G(x) = \int_{g(x)}^{h(x)} f(x,t) \, dt$$

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Bsp.: $G(x) = \int_{1+x}^{e^x} \cos x \; z \, dz$ $\quad e^x = h(x), \quad 1+x = g(x), \quad \cos x \; z = f(x,z)$

$= \cos x \; \frac{1}{2} z^2 \Big|_{z=1+x}^{z=e^x}$

$= \frac{1}{2} \cos x \left\{ e^{2x} - (1+x)^2 \right\}$ . Damit

$G'(x) = -\frac{1}{2} \sin x \left\{ e^{2x} - (1+x)^2 \right\}$

$\qquad + \frac{1}{2} \cos x \left\{ 2 e^{2x} - 2(1+x) \right\}$

$= \int_{1+x}^{e^x} \frac{d}{dx} f(x,z)\, dz + f(x, h(x))\, h'(x)$

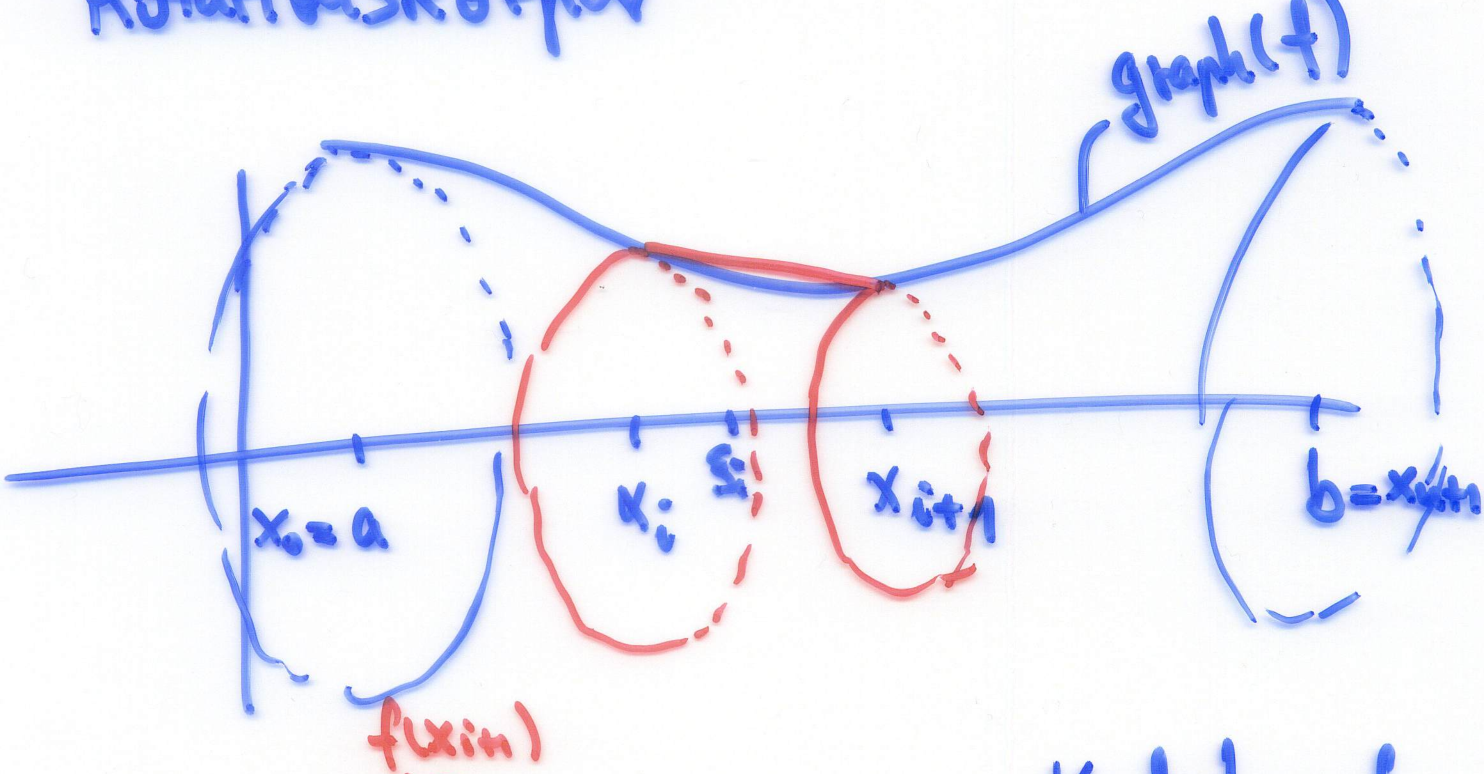
$\qquad - f(x, g(x))\, g'(x)$

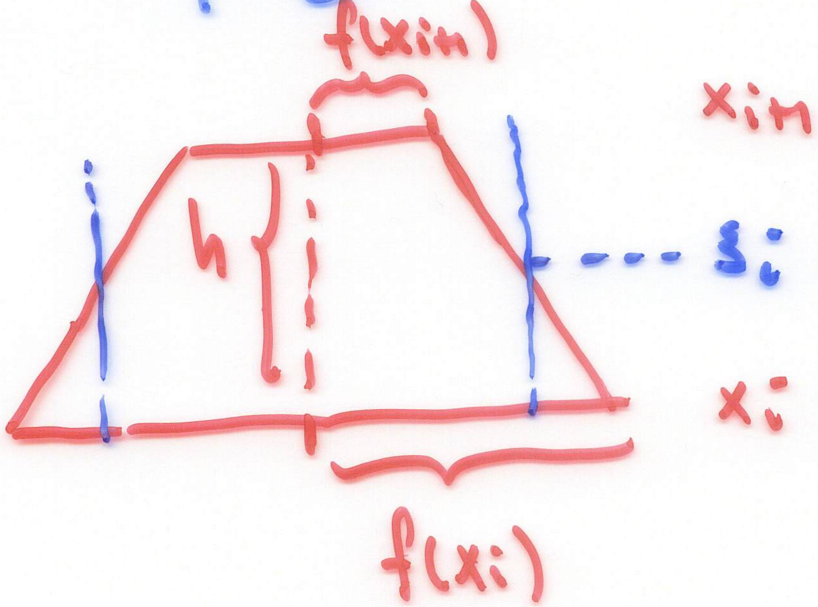
Verifikation durch Einsetzprobe

Leibniz Regel

$$\frac{d}{dx} \int_{g(x)}^{h(x)} f(x,t)\, dt = \int_{g(x)}^{h(x)} \frac{d}{dx} f(x,t)\, dt + f(x, h(x))\, h'(x) - f(x, g(x))\, g'(x)$$

# Rotationskörper




Kegelstumpf

$$V = \pi f(\xi_i)^2 h$$

$h = x_{i+1} - x_i$

$i = 0, \dots, n$

Volumen Rotationskörper

$$\approx \pi \sum_{k=0}^{n} f(\xi_k)^2 \underbrace{h}_{x_{k+1}-x_k} \;\hat{=}\; \text{Riemann-Summe von } \pi \int_a^b f^2(x)\,dx$$

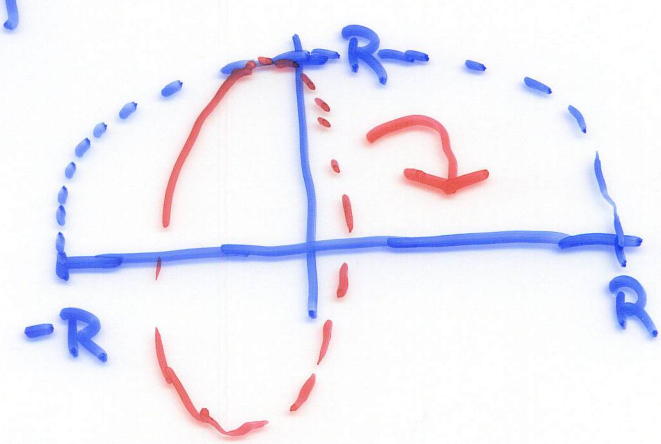
<u>Merke</u>: $f : [a,b] \to \mathbb{R}_0^+$

Rotationskörper hat Volumen

$$V = \pi \int_a^b f^2(x)\, dx$$

<u>Bsp</u>: Kugel - Volumen im $\mathbb{R}^3$

$$f(x) := R \sqrt{1 - \left(\frac{x}{R}\right)^2}$$

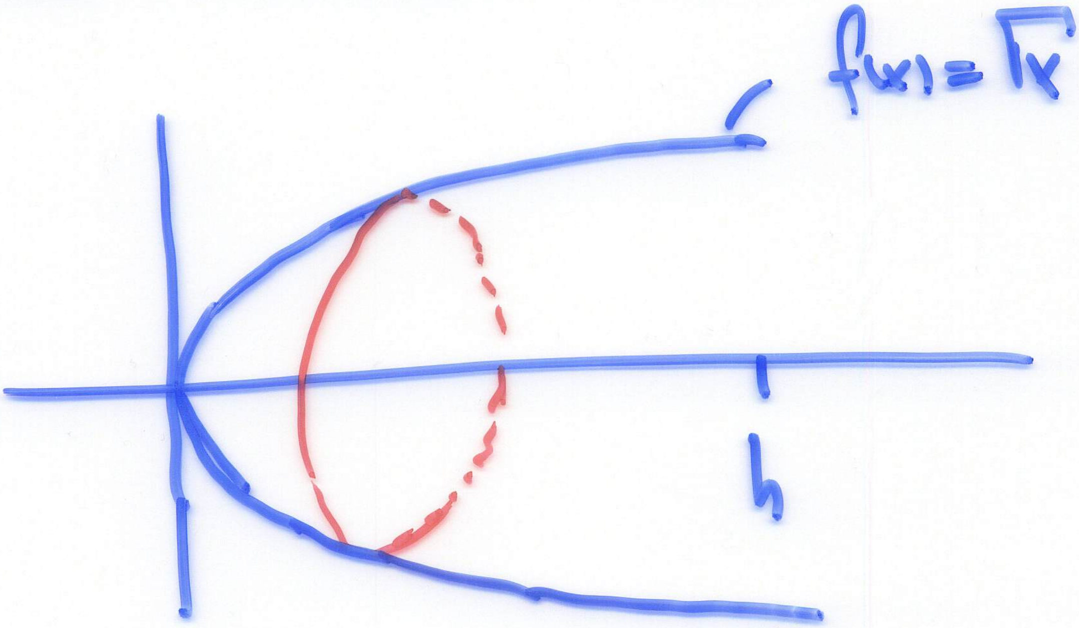



$$V = \pi R^2 \int_{-R}^{R} \sqrt{1 - \left(\frac{x}{R}\right)^2}^{\,2} dx$$

$$= \pi R^2 \int_{-R}^{R} 1 - \left(\frac{x}{R}\right)^2 dx$$

$$= \pi R^2 \left( x - \frac{x^3}{3R^2} \right) \Big|_{x=-R}^{x=R} = \frac{4}{3} \pi R^3$$

Anwendung: Bestimme Höhe $h$ eines Sektkelches mit Fassungsvermögen 0,1 l und Wurzel-Form

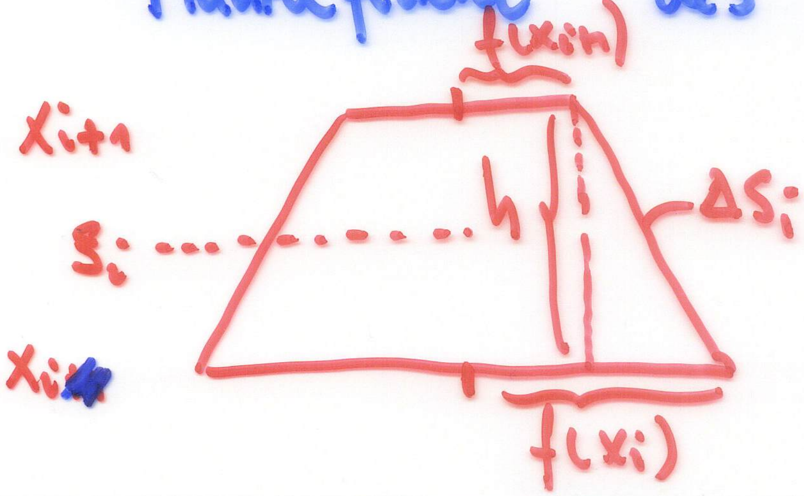



$$V = \pi \int_0^h f^2(x)\,dx = \pi \int_0^h x\,dx$$

$$= \frac{1}{2}\pi h^2 \overset{!}{=} 100\ \text{cm}^3$$

$$\mapsto h = \dots$$

Mantelfläche des Rotationskörpers




$$(\Delta s_i)^2 = h^2 + [f(x_{i+1}) - f(x_i)]^2$$

$$\overset{\text{MWS}}{=} h^2 + f'(\xi_i)^2 h^2$$

Damit

$$\Delta s_i = h\sqrt{1+f'(\xi_i)^2}$$

$$\Rightarrow \sigma_{\text{Kugelstumpf}} \approx 2\pi f(\xi_i)\sqrt{1+f'(\xi_i)^2}\, h$$

$$\rightarrow \sigma_{\substack{\text{Rotations}\\ \text{Körper}}} \approx 2\pi \sum_{k=0}^{n} f(\xi_i)\sqrt{1+f'(\xi_i)^2}\, h$$

$$\underset{\text{Summe}}{\overset{\text{Riemann}}{=}} \quad 2\pi \int_a^b f(x)\sqrt{1+f'(x)^2}\, dx$$

Bsp: Oberfläche Kugel mit Radius R

$$f(x) = R\sqrt{1-\left(\frac{x}{R}\right)^2} \qquad f'(x) = \frac{-\frac{x}{R}}{\sqrt{1-\left(\frac{x}{R}\right)^2}}$$

$$\rightarrow f(x)\sqrt{1+f'(x)^2} = R$$

$$\rightarrow \sigma_{\text{Kugel}} = 2\pi \int_{-R}^{R} R\, dx = 2\pi R\, x\Big|_{-R}^{R} = 4\pi R^2$$