Produkt Integration (1)

$$F(x) := f(x)\,g(x)$$

$$F'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$= (f(x)\,g(x))'$$

$$F(x)\Big|_a^b = f(x)\,g(x)\Big|_a^b$$

$$= \int (f(x)g(x))'\,dx$$

$$= \int f'(x)\,g(x)\,dx + \int f(x)g'(x)\,dx$$

"$\int f'g = fg - \int fg'$"

Herleitung genau wie Substitutionsregel

$$F(x) = f(\varphi(x)) \rightarrow F'(x) = f'(\varphi(x))\varphi'(x)$$

$$\Rightarrow \int_a^b f(\varphi(x))\varphi'(x) = F(\varphi(x))\Big|_a^b$$

Bsp.e Substitution ②

i) $\int_a^b f(t+c_1)\,dt \underset{dz=dt}{\overset{z=t+c_1}{=}} \int_{a+c_1}^{b+c_1} f(z)\,dz$

ii) $\int_a^b f(c_1 t)\,dt \underset{dz=c_1 dt}{\overset{z=c_1 t}{=}} \int_{c_1 a}^{c_1 b} \frac{1}{c_1} f(z)\,dz$

iii) $\int_a^b t\, f(t^2)\,dt \underset{dz=2t\,dt}{\overset{z=t^2}{=}} \int_{a^2}^{b^2} \frac{1}{2} f(z)\,dz$

$\frac{f'(t)}{f(t)}$

iv.) $\int_a^b \frac{f'(t)}{f(t)}\,dt \underset{dz=f'(t)\,dt}{\overset{z=f(t)}{=}} \int_{f(a)}^{f(b)} \frac{1}{z}\,dz$

$= \ln|z| \Big|_{f(a)}^{f(b)} = \ln|f(t)| \Big|_a^b$

v.) Bsp zu iv).

$\int_a^b \tan t\,dt = \int_a^b \frac{\sin t}{\cos t}\,dt = -\int_a^b \frac{-\sin t}{\cos t}\,dt$

140408 3

$$-\int_a^b \frac{-\sin t}{\cos t}\,dt \overset{iv)}{=} -\ln\cos t\Big|_a^b$$

für $[a,b] \subset (-\frac{\pi}{2}, \frac{\pi}{2})$

vi) $\int_a^b \sqrt{1-x^2}\,dx = ?$ mit $-1<a<b<1$

$$\overset{x=\sin t}{=} \int_u^v \underbrace{\sqrt{1-\sin^2 t}}_{\cos t}\,dt \quad \text{mit} \quad \begin{matrix} u=\arcsin a \\ v=\arcsin b \end{matrix}$$

$$= \int_u^v \cos^2 t\,dt = \int_u^v \frac{1}{2}(\cos 2t + 1)\,dt$$

$$= \frac{1}{4}\sin 2t\Big|_u^v + \frac{1}{2}t\Big|_u^v$$

$$= \frac{1}{2}\sin t\sqrt{1-\sin^2 t}\Big|_u^v + \frac{1}{2}t\Big|_u^v$$

benutzt: $\sin 2t = 2\sin t\cos t$

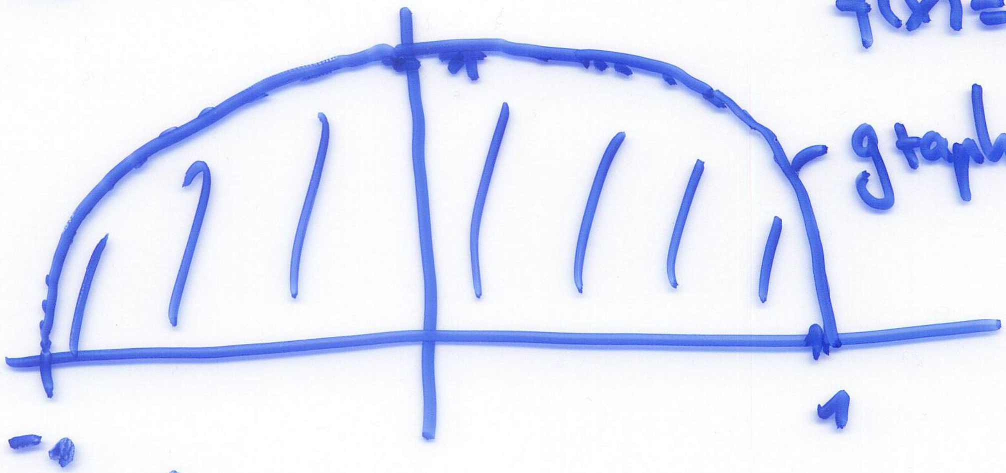
$= 2\sin t\sqrt{1-\sin^2 t}$

$$\overset{t=\arcsin x}{=} \frac{1}{2}\left[\arcsin x\Big|_a^b + x\sqrt{1-x^2}\Big|_a^b\right]$$

14.04.08 ④

Folgerung aus vi)

$f(x) = \sqrt{1-x^2}$



$$/// = \int_{-1}^{1} \sqrt{1-x^2}\, dx = \frac{\pi}{2}$$

$\overset{\text{vi)}}{=}$

$$\frac{1}{2}\left[\arcsin(1) - \arcsin(-1) + 0\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] = \frac{\pi}{2}$$

Produktintegration $a, b > 0$

i) $\int_a^b \ln x \, dx = \int_a^b \underset{f'}{1} \underset{g}{\ln x}\, dx$

$$= \underset{f\ \ g}{x \ln x}\Big|_a^b - \int_a^b x \cdot \frac{1}{x}\, dx$$

$$= (x \ln x - x)\Big|_a^b$$

ii) $\int \sin^2 x\,dx = \int \underset{f'}{\sin x}\,\underset{g}{\sin x}\,dx$

$= -\cos x \sin x - \int (-\cos x)\cos x\,dx$

$= -\cos x \sin x + \int 1 - \sin^2 x\,dx$

$= -\cos x \sin x + x - \int \sin^2 x\,dx$

$\rightarrow 2\int \sin^2 x\,dx = x - \cos x \sin x$

d.h. $\int \sin^2 x\,dx = \frac{1}{2}(x - \cos x \sin x)$.

Integration rationaler Funktionen

Bsp $\int \frac{1}{1-x^2}\,dx = ?$

Es gilt $1 - x^2 = (1+x)(1-x)$

Damit

$$\frac{1}{1-x^2} = \frac{a}{1+x} + \frac{b}{1-x} \qquad a = ? \quad b = ?$$

$\rightarrow \int \frac{1}{1-x^2}\,dx = a\int \frac{1}{1+x}\,dx + b\int \frac{1}{1-x}\,dx$

Es ist $\int \frac{1}{1+x}\,dx = \ln|1+x| + C_1$ — Integrationskonstant

$\int \frac{1}{1-x}\,dx = -\ln|1-x| + C_1$

Also: $\int \frac{1}{1-x^2}\,dx = a \ln|1+x|$

$\mp b \ln|1-x| + C_1$

Bestimmen a, b

$$\frac{1}{1-x^2} = \frac{1}{(1+x)(1-x)} = \frac{a}{1+x} + \frac{b}{1-x}$$

$$\rightarrow \quad a = b = \frac{1}{2}$$

$$\rightarrow \quad \int \frac{1}{1-x^2}\,dx = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right| + C_1$$

Allgemeine Situation

$$r(x) = \frac{p_n(x)}{q_m(x)} \quad \text{mit}$$

$$p_n(x) = a_0 + a_1 x + \dots + a_n x^n, \; a_n \neq 0$$

$$q_m(x) = b_0 + b_1 x + \dots + b_m x^m, \; b_m \neq 0$$

$$\int r(x)\,dx = \int \frac{p_n(x)}{q_m(x)}\,dx = ?$$

mit $\deg p_n = n$, $\deg q_m = m$

deg von degree, deutsch: grad

<u>Hier:</u> $n < m$ vorausgesetzt. Dann ist $r(x)$ echt gebrochen rationale Funktion

$n \geq m$: Polynomdivision

$$\frac{p_n}{q_m} = s_{n-m} + r$$

mit s_{n-m} Polynom vom Grade $n-m$.

und r echt gebrochen rational

$n < m$: $\frac{p_n}{q_m}$ darstellbar mittels Partialbruchzerlegung

Bsp:

$$\int \frac{x+1}{x^4-x}\,dx = ? \qquad r(x) = \frac{p(x)}{q(x)}$$

$p(x) = p_1(x) = 1+x \quad \deg p_1 = 1$

$q(x) = q_4(x) = x^4-x \quad \deg q_4 = 4$

$q(x) = x^4 - x = x(x-1)(x^2+x+1)$

$$\rightarrow r(x) \overset{PBZ}{=} \frac{a}{x} + \frac{b}{x-1} + \frac{cx+d}{x^2+x+1}$$

$$\| $$

$$\frac{x+1}{x(x-1)(x^2+x+1)}$$

$\rightarrow \quad a = -1 \quad b = \frac{2}{3} \quad c = \frac{1}{3} \quad d = -\frac{1}{3}$

$$\rightarrow \int \frac{x+1}{x^4-x}\,dx = -\ln|x| + \frac{2}{3}\ln|x-1| + \frac{1}{3}\underbrace{\int \frac{x-1}{x^2+x+1}\,dx}$$

IV) Substitution bzw ← Formelsammlung