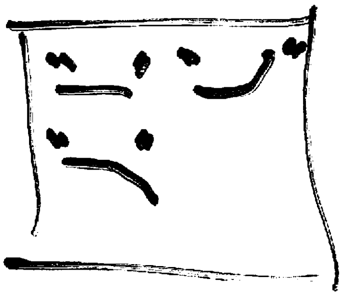
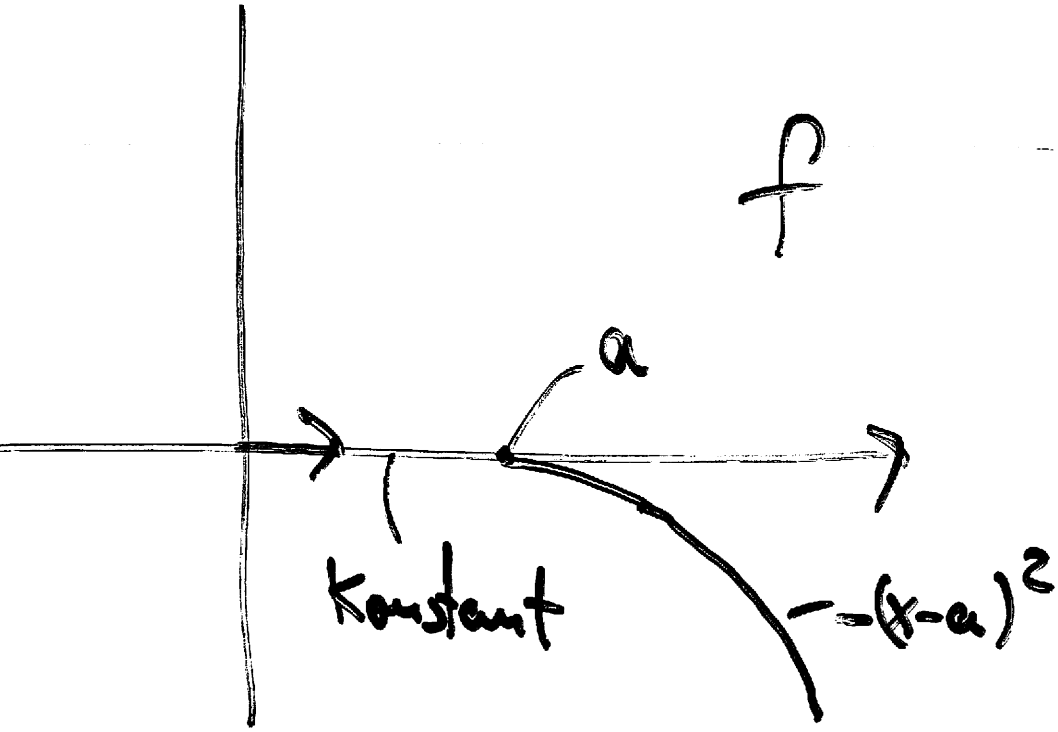
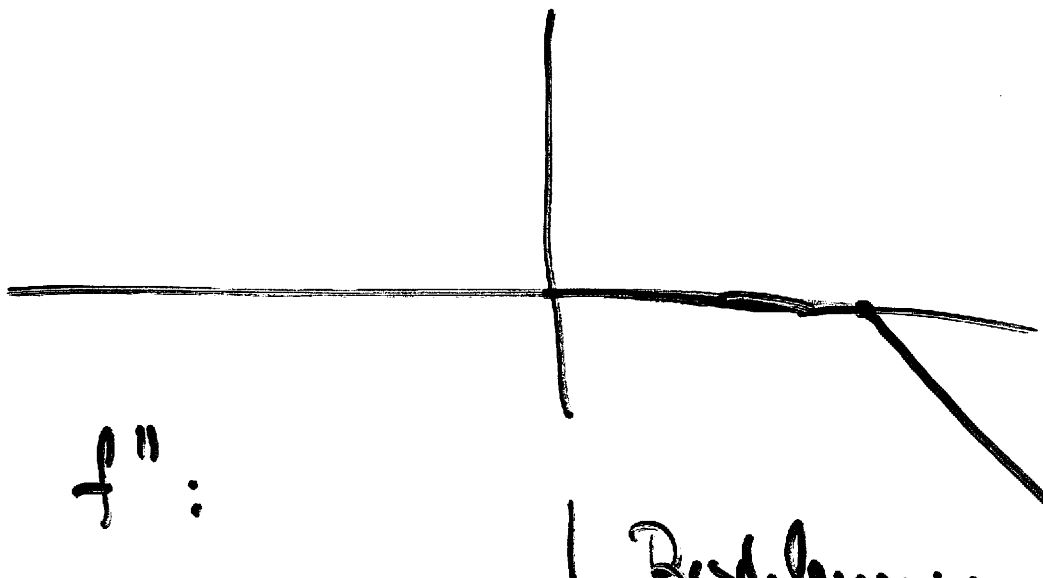
181207 ①

Differenzierbarkeit & das lineare Modell



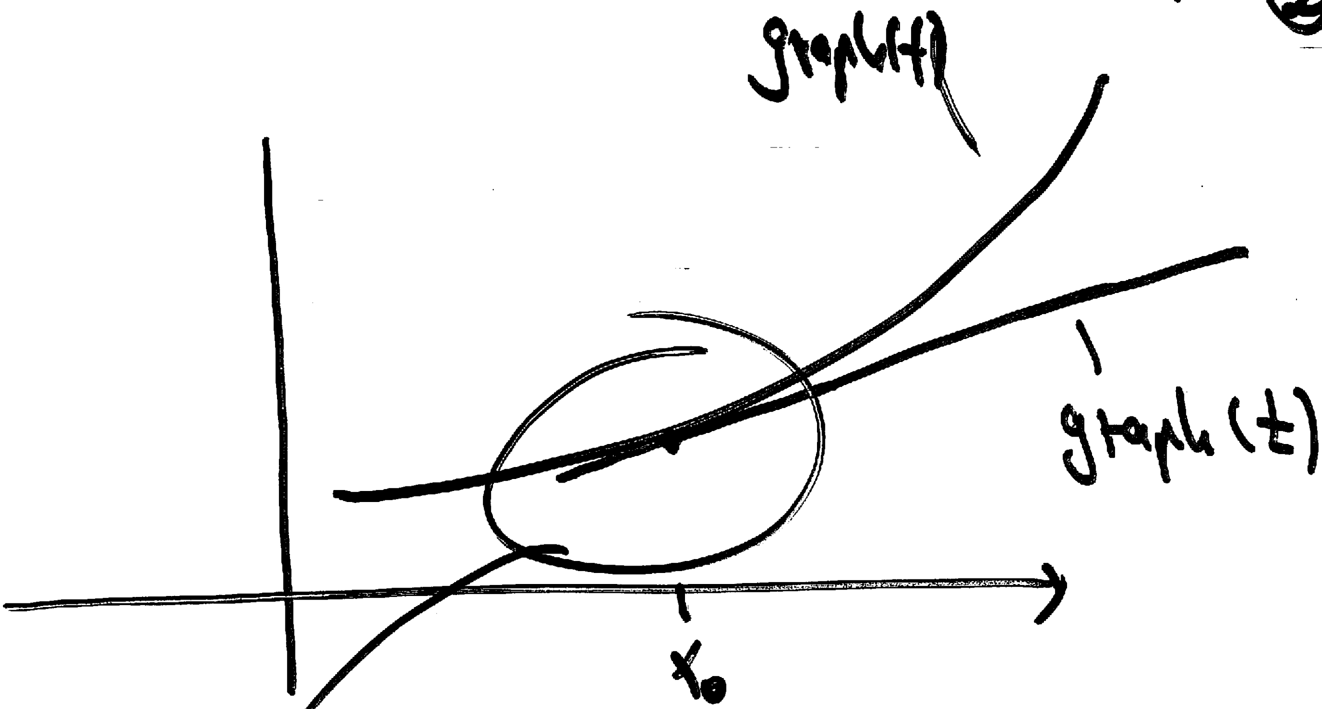


f':



f'':

Beschleunigung springt!



$$t(x) = f(x_0) + f'(x_0)(x - x_0)$$

Zoom

$(x_0, f(x_0))$

$t(x) - f(x)$ "klein" in der Nähe von x_0

Sei f in x_0 diffbar. Dann

$$\lim_{x \to x_0} \left[\frac{f(x) - f(x_0)}{x - x_0} - f'(x_0) \frac{x - x_0}{x - x_0} \right] = 0$$

d.h. mit

$$k(x) := f(x) - f(x_0) - f'(x_0)(x-x_0)$$

gilt

$$\lim_{x \to x_0} \frac{k(x)}{x-x_0} = 0$$

<u>Merke</u>: $k(x)$ konvergiert schneller gegen 0 für $x \to x_0$ als $x-x_0$

Beachte

$$f(x) = \underline{f(x_0) + f'(x_0)(x-x_0)}$$

$$+ f(x) - f(x_0) - f'(x_0)(x-x_0)$$

$$= t(x) + \underbrace{k(x)}_{\text{„klein“ in der Nähe von } x_0}$$

Bsp i.) $f(x) = \ln x \quad x_0 = e$

$$t(x) = f(x_0) + f'(x_0)(x - x_0)$$
$$= \ln e + \frac{1}{e}(x - e)$$

$$f(3) \approx t(3) = 1 + \frac{1}{e}(3 - e)$$
$$\parallel \qquad\qquad = 1.10364$$
$$1.09861$$

$$\Longrightarrow \quad f(3) - t(3) = -0.005$$

ii) $f(x) = \sin x, \quad x_0 = 0$

$$t(x) = 0 + 1(x - 0) = x$$

Damit

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{t(x) + R(x)}{x}$$
$$= \lim_{x \to 0} \frac{x}{x} + \lim_{x \to 0} \frac{R(x)}{x} = 1$$

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iii) $f(x) = \cos x\ ,\ x_0 = 0$

$$t(x) = 1 - 0\,(x-0) = 1$$

Damit

$$\lim_{x\to 0} \frac{1-\cos x}{x} = \lim_{x\to 0} \frac{1 - t(x) - k(x)}{x}$$

$$= \underbrace{\lim_{x\to 0} \frac{0}{x}}_{=0} - \underbrace{\lim_{x\to 0} \frac{k(x)}{x}}_{=0}$$

$$= 0$$

iv). $f(x) = e^x,\quad x_0 = 0$

$$t(x) = e^0 + e^0(x-0) = 1+x$$

$f: I \to \mathbb{R}$ in $x_0 \in I$ diffbar und bei x_0 nehme f Infimum oder Supremum an.

Ist dann $x_0 \in (a,b)$, so gilt

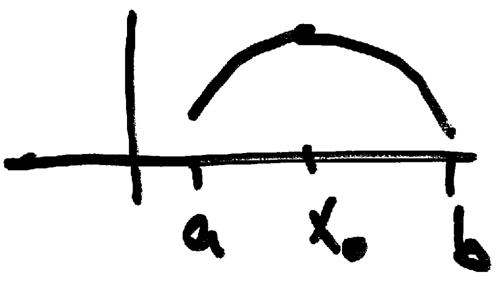
$$f'(x_0) = 0$$

Nachweis: bei x_0 Supremum,

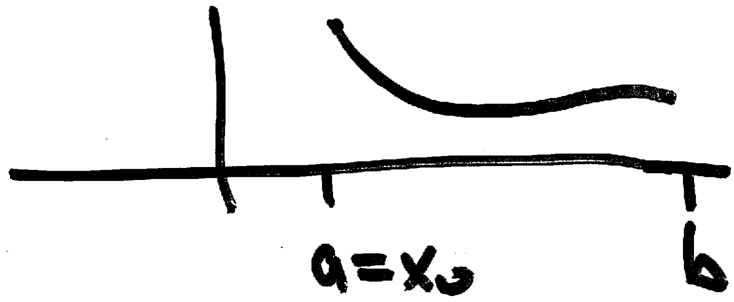
d.h. $f(x) \leq f(x_0) \quad \forall x \in I$

$$\Rightarrow \quad \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} \begin{cases} \geq 0, & x < x_0 \\ \leq 0, & x > x_0 \end{cases}$$

$$\Rightarrow \quad f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0} = 0$$



$x_0 = a$:



$$0 \geq \frac{f(x_0) - f(x)}{x_0 - x} \xrightarrow{x \to x_0} f'(x_0)$$

Analog : $x_0 = b$: $f'(x_0) \geq 0$

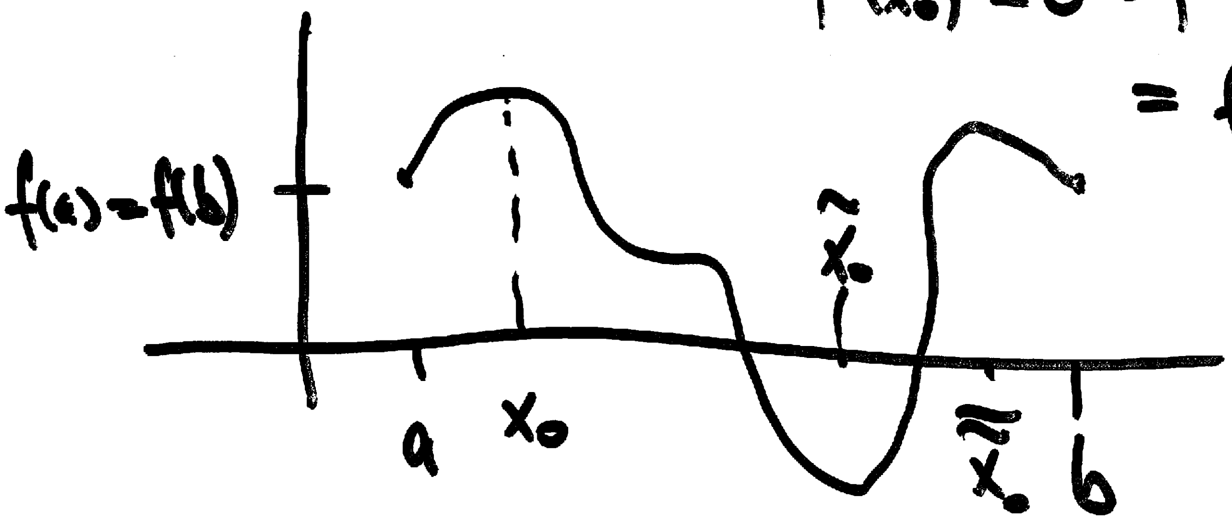
Infimum bei x_0 : $x_0 \in (a,b)$: $f'(x_0) = 0$

$x_0 = a$: $f'(x_0) \geq 0$

$x_0 = b$: $f'(x_0) \leq 0$

Satz von Rolle

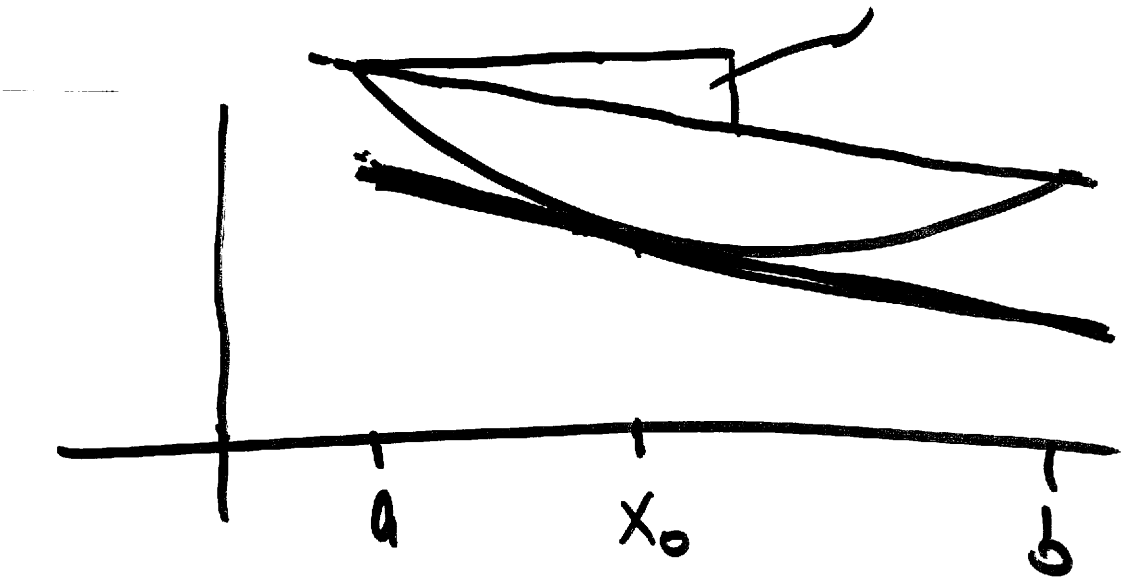
$$f'(x_0) = 0 = f'(\tilde{x}_0) = f'(\tilde{\tilde{x}}_0)$$



MWS f diffbar auf [a,b]

Dann ex $x_0 \in (a,b)$:

$$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$



Nachweis: Wende Satz von Rolle auf

$$g(x) = f(x) - f(a) - \frac{f(b)-f(a)}{b-a}(x-a)$$

$$g(a) = 0 = g(b)$$

$$\Rightarrow \quad 0 = g'(x_0) = f'(x_0) - \frac{f(b)-f(a)}{b-a},$$

d.h. $$f'(x_0) = \frac{f(b)-f(a)}{b-a}$$

Anwendung: Abschätzung von Fehlern

$$\frac{f(x) - f(x_1)}{x - x_1} \overset{\text{MWS}}{=} f'(x_0)$$

mit $x_0 \in (x, x_1)$

$$\Rightarrow \quad |f(x) - f(x_1)| \le |f'(x_0)|\,|x - x_1|$$

d.h.

$$\max_{|x-x_1|\le\delta} |f(x)-f(x_1)| \le \max_{|x-x_1|\le\delta} |f'(x)|\,\delta$$