A-Posteriori Error Estimators for RBM Applied to Quadratic Non-Linear PPDEs (Involving Non-Affine Coefficient Functions)

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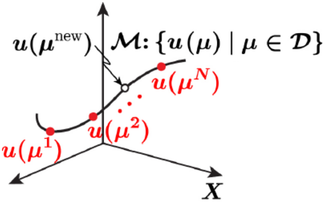
(Joint work with Claudio Canuto, Karsten Urban)

Reduced-Basis Methods for solving PPDEs **rapidly**, **repeatedly** and **reliably**.



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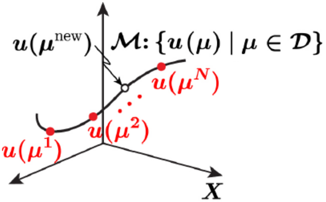

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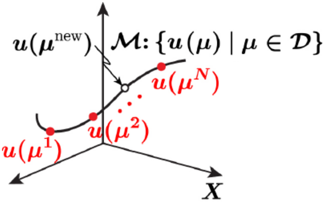

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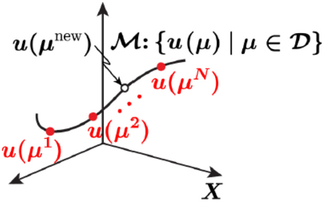

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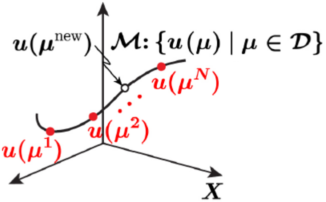

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- Availability of **a-posteriori error estimators**
  1. ensures reliability;
  2. allows for optimal basis selection.
- Efficient treatment of **output functionals**.
- **Offline/Online Decomposition** allows for a $\mathcal{N}$–independent online-stage.

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Compute

- $u_j := u(\mu_j)$, $1 \leq j \leq n$; $K = \big((u_i, u_j)_X\big)_{1 \leq i,j \leq n}$;
- $\xi_i = \frac{1}{\sqrt{\lambda_i}} \sum_{j=1}^{n} (v_i)_j u_j$, $1 \leq i \leq N$, where λ_i is the i-th largest eigenvalue of K and v_i is the corresponding eigenvector.

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Then, it is proven that

1. $(\xi_i, \xi_j)_X = \delta_{ij}$, $1 \leq i, j \leq n$ and
2. $\int_{\mathcal{D}} \left\| u(\mu) - \sum_{i=1}^{N} (u(\mu), \xi_i)_X \, \xi_i \right\|_X^2 d\mu = \sum_{i=N+1}^{n} \lambda_i$.

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Then for $N = 1, 2, \ldots$

1. compute
$$\mu^* := \arg\max_{\mu \in \Xi} \Delta(\mu);$$
2. if $\Delta(\mu^*) > \varepsilon$, update $S^{N+1} := S^N \cup \{\mu^*\}$ and continue;
3. stop,

where $\Delta(\mu)$ is a **rapidly evaluable**, **reliable a-posteriori** error estimator for $\|u(\mu) - \widehat{u}(\mu)\|_X$.

Application

For $\mu \in \mathcal{D} := [0, \frac{\pi}{2}]$ we aim in solving

$$\begin{cases} -a\Delta u + (\underline{b} \cdot \nabla u)\, u + cu^2 = 0 \ , & \text{in } \Omega(\mu), \\ u = 0 \ , & \text{on } \partial B(\mu), \\ u = g \ , & \text{on } \Gamma_D, \\ \frac{\partial u}{\partial n} = 0 \ , & \text{on } \Gamma_N. \end{cases}$$

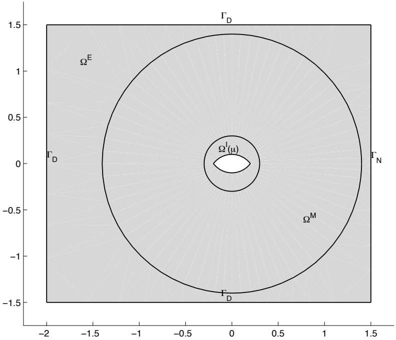

Figure: Geometry for one blade.

Mapping to a **reference situation** yields:

Find $\check{u}(\mu) \in u_0 + X$, s.t. $\forall \check{v} \in X$:

$$\int_{\check{\Omega}} \left(\underline{\underline{\alpha}}(\cdot;\mu)\nabla\check{u}\right) \cdot \nabla\check{v}\, d\check{\Omega} + \int_{\check{\Omega}} \left(\underline{\beta}(\cdot;\mu)\cdot\nabla\check{u}\right) \check{u}\check{v}\, d\check{\Omega} + \int_{\check{\Omega}} \gamma(\cdot;\mu)\check{u}^2\check{v}\, d\check{\Omega} = 0,$$

where

- $X := \{v \in H^1(\check{\Omega}) : v = 0 \text{ on } \Gamma_D \cup \partial\check{B}\}$;
- $u_0 \in H^1(\Omega_E) : u_0 = g \text{ on } \Gamma_D, u_0 = 0 \text{ on } \partial\Omega_E \setminus \Gamma_D$.

Primal Problem (Generalization of [?] (K. Veroy, A.T. Patera; 2005)):

$$\begin{cases} \text{For } \mu \in \mathcal{D} \subset \mathbb{R}^p \text{ find } u(\mu) \in X, \text{ s.t. for all } v \in X: \\ g(u(\mu), v; h(\mu)) := a(u(\mu), v; h_a(\mu)) + b(u(\mu), u(\mu), v; h_b(\mu)) - f(v) = 0, \end{cases}$$

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where

- $\Omega \subset \mathbb{R}^n$, $X \subset X^e \subset H^1(\Omega)$;
- $h(\mu) := \{h_a(\mu), h_b(\mu)\}$, $h_i(\mu) := h_i(\cdot; \mu)$, $h_i \in L^\infty(\Omega) \times C^1(\mathcal{D})$;
- $a(w, v; h_a)$ is linear in $w, v \in X$ and $h_a \in L^\infty(\Omega)$;
- $b(w, z, v; h_b)$ is linear in $w, z, v \in X$ and $h_b \in L^\infty(\Omega)$;
- f is a bounded, linear functional in X.

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- f is a bounded, linear functional in X.

Output of Interest:

- $s(\mu) := \ell(u(\mu))$, where ℓ is a bounded, linear functional in X.

Frechét derivative $dg(\cdot, \cdot; h)[z] : X \times X \to \mathbb{R}$ at $z \in X$ is

$$dg(w, v; h)[z] := a(w, v; h_a) + \underbrace{b(w, z, v; h_b) + b(z, w, v; h_b)}_{=: db(w,v;h_b)[z]}.$$

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Inf-sup parameter and **continuity constant** (for fixed $z \in X$):

$$\beta(z;h) := \inf_{w\in X} \sup_{v\in X} \frac{dg(w,v;h)[z]}{\|w\|_X \|v\|_X},$$

$$\gamma(z;h) := \sup_{w\in X} \sup_{v\in X} \frac{dg(w,v;h)[z]}{\|w\|_X \|v\|_X}.$$

For **well-posedness** $\{u(\mu), \mu \in \mathcal{D}\}$ is required to be a **non-singular (isolated) branch**, thus we pose

Assumption

There is $\beta_0 > 0$, *s.t.* $\forall \mu \in \mathcal{D}$: $\beta(u(\mu); h(\mu)) \geq \beta_0$.

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Furthermore, we pose

Assumption

$\forall \mu \in \mathcal{D}$ *there is* $0 \leq \rho_a, \rho_b < \infty$, *s.t.* $\forall w, z, v \in X$:

$$\begin{aligned} |a(w, v; h_a)| &\leq \rho_a \|w\|_X \|v\|_X \|h_a\|_{L^\infty(\Omega)}, \\ |b(w, z, v; h_b)| &\leq \rho_b \|w\|_X \|z\|_X \|v\|_X \|h_b\|_{L^\infty(\Omega)}. \end{aligned}$$

In the sequel: $\rho_i(\mu) := \rho_i \|h_i(\mu)\|_{L^\infty(\Omega)}$, $i \in \{a, b\}$.

For $1 \leq N \leq N^{\max}$, let

- $S^N := \{\mu_n \in \mathcal{D}, 1 \leq n \leq N\}$ be a **nested set of parameter samples**, and
- $W^N := \text{span}\{\xi_n := u(\mu_n), 1 \leq n \leq N\}$ the associated **reduced-basis Lagrangian space**.

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Primal Reduced-Basis Problem:

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \widehat{u}(\mu) \in W^N, \text{ s.t. for all } v \in W^N: \\ g(\widehat{u}(\mu), v; \widehat{h}(\mu)) = 0, \end{cases}$$

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where we have replaced $h(\mu) := \{h_a(\mu), h_b(\mu)\}$ by $\widehat{h}(\mu) := \{\widehat{h}_a(\mu), \widehat{h}_b(\mu)\}$ and

$$\widehat{h}_i(\mu) := \sum_{m=1}^{M_i} \varphi^i_m(\mu) q^i_m, \quad i \in \{a, b\},$$

obtained by **Empirical Interpolation Method (EIM)**
(c.p. M. Barrault, Y. Maday, N.C. Nguyen, A.T. Patera; 2004).

Solved iteratively by **Newton's method**:

- Initial guess: $\widehat{u}^{(0)}(\mu) \in W^N$.
- For $k = 0, 1, 2, \ldots$:
  1. Find $\delta^{(k)}(\mu) \in W^N$, s.t. $\forall v \in W^N$:

$$dg(\delta^{(k)}(\mu), v; \widehat{h}(\mu))[\widehat{u}^{(k)}(\mu)] = -g(\widehat{u}^{(k)}(\mu), v; \widehat{h}(\mu));$$

  2. Update $\widehat{u}^{(k)}(\mu)$ by $\widehat{u}^{(k+1)}(\mu) = \widehat{u}^{(k)}(\mu) + \delta^{(k)}(\mu)$.

For the **offline/online decomposition**, let

$$
\begin{aligned}
F &:= \left(f\left(\xi_i\right)\right)_i, \\
A_m &:= \left(a\left(\xi_j, \xi_i; q_m^a\right)\right)_{i,j}, && 1 \leq m \leq M_a, \\
B_m^n &:= \left(b\left(\xi_j, \xi_n, \xi_i; q_m^b\right)\right)_{i,j}, && 1 \leq m \leq M_b, 1 \leq n \leq N, \\
dB_m^n &:= \left(db\left(\xi_j, \xi_i; q_m^b\right)\left[\xi_n\right]\right)_{i,j}, && 1 \leq m \leq M_b, 1 \leq n \leq N.
\end{aligned}
$$

The coefficients $\delta_n^{(k)}(\mu)$, $1 \leq n \leq N$, can be obtained by solving

$$
\begin{aligned}
&\left[\sum_{m=1}^{M_a} \varphi_m^a(\mu) A_m + \sum_{n=1}^{N} \widehat{u}_n^{(k)}(\mu) \sum_{m=1}^{M_b} \varphi_m^b(\mu) dB_m^n\right] \underline{\delta}^{(k)}(\mu) \\
&\quad = -\left[\sum_{m=1}^{M_a} \varphi_m^a(\mu) A_m + \sum_{n=1}^{N} \widehat{u}_n^{(k)}(\mu) \sum_{m=1}^{M_b} \varphi_m^b(\mu) B_m^n\right] \underline{\widehat{u}}^{(k)}(\mu) + F,
\end{aligned}
$$

within $\mathcal{O}\left(M_a N^2\right) + \mathcal{O}\left(M_b N^3\right) + \mathcal{O}\left(N^3\right)$.

$\rightsquigarrow$ **(online-)complexity independent of $\mathcal{N} \gg N$.**

Improvement of the output approximation $\widehat{s}_1(\mu) := \ell\left(\widehat{u}(\mu)\right)$ necessitates invoking the **Dual Problem**:

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \psi(\mu) \in X, \text{ s.t. for all } v \in X\text{:} \\ dg\left(v, \psi(\mu); \widehat{u}(\mu) + \dfrac{1}{2}e(\mu); h(\mu)\right) = -\ell(v), \end{cases}$$

where $e(\mu) := u(\mu) - \widehat{u}(\mu)$.

$\rightsquigarrow$ This is a **linear** problem.

For $1 \leq \widetilde{N} \leq \widetilde{N}^{\max}$, let

- $\widetilde{S}^{\widetilde{N}} := \{\mu_n \in \mathcal{D}, 1 \leq n \leq \widetilde{N}\}$ be a **nested set of parameter samples**, and
- $\widetilde{W}^{\widetilde{N}} := \mathrm{span}\{\widetilde{\xi}_n := \psi^{N^{\max}}(\mu_n), 1 \leq n \leq \widetilde{N}\}$ the associated **reduced-basis Lagrangian space**.

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Dual Reduced-Basis Problem:

$$\begin{cases} \text{For } \mu \in \mathcal{D} \text{ find } \widehat{\psi}(\mu) \in \widetilde{W}^{\widetilde{N}}, \text{ s.t. for all } v \in \widetilde{W}^{\widetilde{N}}: \\ dg(v, \widehat{\psi}(\mu); \widehat{u}(\mu); \widehat{h}(\mu)) = -\ell(v). \end{cases}$$

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(Online-)complexity is $\mathcal{O}\left(M_a \widetilde{N}^2\right) + \mathcal{O}\left(M_b N \widetilde{N}^2\right) + \mathcal{O}\left(\widetilde{N}^3\right)$.
$\rightsquigarrow$ comparable to **one** Newton iteration (for solving the primal reduced-basis problem).

Existence, Uniqueness and Well-Posedness

First, we define a **proximity indicator** (the key parameter for the **BRR** theory):

$$\tau(\mu) \quad := \quad 4\rho_b(\mu)\big(\widehat{\beta}(\mu)\big)^{-2}\big(R(\mu)+E(\mu)\big),$$

where (for $v \in X$ and $\mu \in \mathcal{D}$)

$$\begin{aligned} R(v;\mu) &:= g\left(\widehat{u}(\mu), v; \widehat{h}(\mu)\right), & R(\mu) &:= \|R(\cdot;\mu)\|_{X'}\,, \\ E(v;\mu) &:= g\left(\widehat{u}(\mu), v; h(\mu)-\widehat{h}(\mu)\right), & E(\mu) &:= \|E(\cdot;\mu)\|_{X'}\,, \end{aligned}$$

and $0 < \widehat{\beta}(\mu) \leq \beta(\widehat{u}(\mu); h(\mu))$ for all $\mu \in \mathcal{D}$.

Corollary

For $\tau(\mu) \leq \frac{1}{2}$ *it is true, that* $\beta(u(\mu); h(\mu)) \geq \widehat{\beta}(\mu)/\sqrt{2} > 0$.

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Immediate consequences:

- The primal problem is well-posed
  ($\{u(\mu), \mu \in \mathcal{D}\}$ is a non-singular (isolated) branch).
- The dual problem is well-posed and possesses a unique solution.

Proposition

For $\tau(\mu) < 1$ there is a unique $u(\mu)$ such that

- $u(\mu) \in \mathcal{B}\left(\widehat{u}(\mu), \widehat{\beta}(\mu)\left(2\rho_b(\mu)\right)^{-1}\right)$,
  where $\mathcal{B}(z, r) := \{v \in X : \|v - z\|_X < r\}$,
- $\|u(\mu) - \widehat{u}(\mu)\|_X \leq \Delta(\mu) := \widehat{\beta}(\mu)\left(2\rho_b(\mu)\right)^{-1}\left(1 - \sqrt{1 - \tau(\mu)}\right)$.

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Proof: Use **Banach Fixed Point Theorem** for $H^\mu : X \to X$ defined by

$$dg(H^\mu w, v; h(\mu))[\widehat{u}(\mu)] = dg(w, v; h(\mu))[\widehat{u}(\mu)] - g(w, v; h(\mu)),\ \ v \in X.$$

For the **dual problem** we define an **a-posteriori error estimator** by

$$\widetilde{\Delta}(\mu) := \frac{2\left(\widetilde{R}(\mu) + \widetilde{E}(\mu)\right)}{\widehat{\beta}(\mu)(1+\sqrt{1-\tau(\mu)})} + \frac{(1-\sqrt{1-\tau(\mu)})}{(1+\sqrt{1-\tau(\mu)})}\left\|\widehat{\psi}(\mu)\right\|_X,$$

where (similar to the primal problem)

$$\begin{aligned}
\widetilde{R}(\mu) &:= \left\|dg\left(\cdot, \widehat{\psi}(\mu); \widehat{u}(\mu); \widehat{h}(\mu)\right) + \ell(\cdot)\right\|_{X'}, \\
\widetilde{E}(\mu) &:= \left\|dg\left(\cdot, \widehat{\psi}(\mu); \widehat{u}(\mu); h(\mu) - \widehat{h}(\mu)\right)\right\|_{X'}.
\end{aligned}$$

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We may proof that

Corollary

If $\tau(\mu) < 1$, then it holds $\left\|\psi(\mu) - \widehat{\psi}(\mu)\right\|_X \leq \widetilde{\Delta}(\mu)$.

For the **output of interest** we find $\left(\widetilde{e}(\mu) := \psi(\mu) - \widehat{\psi}(\mu)\right)$:

$$s(\mu) - \underbrace{\ell\left(\widehat{u}(\mu)\right)}_{=:\widehat{s}_1(\mu)} \quad = \quad R\left(\widehat{\psi}(\mu); \mu\right) + E\left(\widehat{\psi}(\mu); \mu\right) + g\left(\widehat{u}(\mu), \widetilde{e}(\mu); h(\mu)\right),$$

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or

$$s(\mu) - \Bigg(\underbrace{\ell\left(\widehat{u}(\mu)\right) + R\left(\widehat{\psi}(\mu);\mu\right)}_{=:\widehat{s}_2(\mu)}\Bigg) \quad = \quad E\left(\widehat{\psi}(\mu);\mu\right) + g\left(\widehat{u}(\mu), \widetilde{e}(\mu); h(\mu)\right).$$

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Therefore, for the corresponding **a-posteriori error estimators** we find

Corollary

If $\tau(\mu) < 1$, then it holds

1. $|s(\mu) - \widehat{s}_1(\mu)| \leq \Delta_{s_1}(\mu) := \left|R\left(\widehat{\psi}(\mu);\mu\right)\right| + \left|E\left(\widehat{\psi}(\mu);\mu\right)\right| + (R(\mu) + E(\mu))\,\widetilde{\Delta}(\mu);$
2. $|s(\mu) - \widehat{s}_2(\mu)| \leq \Delta_{s_2}(\mu) := \left|E\left(\widehat{\psi}(\mu);\mu\right)\right| + (R(\mu) + E(\mu))\,\widetilde{\Delta}(\mu).$

Summary of quantities:

$\tau(\mu)$: $R(\mu)$, $E(\mu)$, $\widehat{\beta}(\mu)$,

$\Delta(\mu)$: no additional terms,

$\widetilde{\Delta}(\mu)$: $\widetilde{R}(\mu)$, $\widetilde{E}(\mu)$, $\left\|\widehat{\psi}(\mu)\right\|_X$,

$\Delta_s(\mu)$: $R\left(\widehat{\psi}(\mu);\mu\right)$, $E\left(\widehat{\psi}(\mu);\mu\right)$.

(Online-)complexity:

Summary of quantities:

$\tau(\mu)$: $R(\mu)$, $E(\mu)$, $\widehat{\beta}(\mu)$,

$\Delta(\mu)$: no additional terms,

$\widetilde{\Delta}(\mu)$: $\widetilde{R}(\mu)$, $\widetilde{E}(\mu)$, $\left\|\widehat{\psi}(\mu)\right\|_X$,

$\Delta_s(\mu)$: $R\left(\widehat{\psi}(\mu);\mu\right)$, $E\left(\widehat{\psi}(\mu);\mu\right)$.

(Online-)complexity:

$R(\mu)$: $\mathcal{O}\left(M_b^2 N^4\right)$,

$\widetilde{R}(\mu)$: $\mathcal{O}\left(M_b^2 N^2 \widetilde{N}^2\right)$,

Summary of quantities:

$\tau(\mu)$: $R(\mu)$, $E(\mu)$, $\widehat{\beta}(\mu)$,

$\Delta(\mu)$: no additional terms,

$\widetilde{\Delta}(\mu)$: $\widetilde{R}(\mu)$, $\widetilde{E}(\mu)$, $\left\|\widehat{\psi}(\mu)\right\|_X$,

$\Delta_s(\mu)$: $R\left(\widehat{\psi}(\mu);\mu\right)$, $E\left(\widehat{\psi}(\mu);\mu\right)$.

(Online-)complexity:

$R(\mu)$:	$\mathcal{O}\left(M_b^2 N^4\right)$,	$E(\mu)$:	$\mathcal{O}\left(N^4\right)$,
$\widetilde{R}(\mu)$:	$\mathcal{O}\left(M_b^2 N^2 \widetilde{N}^2\right)$,	$\widetilde{E}(\mu)$:	$\mathcal{O}\left(N^2 \widetilde{N}^2\right)$,

Summary of quantities:

$\tau(\mu)$:	$R(\mu)$, $E(\mu)$, $\widehat{\beta}(\mu)$,	$\widetilde{\Delta}(\mu)$:	$\widetilde{R}(\mu)$, $\widetilde{E}(\mu)$, $\left\|\widehat{\psi}(\mu)\right\|_X$,
$\Delta(\mu)$:	no additional terms,	$\Delta_s(\mu)$:	$R\left(\widehat{\psi}(\mu);\mu\right)$, $E\left(\widehat{\psi}(\mu);\mu\right)$.

(Online-)complexity:

$R(\mu)$:	$\mathcal{O}\left(M_b^2 N^4\right)$,	$E(\mu)$:	$\mathcal{O}\left(N^4\right)$,
$\widetilde{R}(\mu)$:	$\mathcal{O}\left(M_b^2 N^2 \widetilde{N}^2\right)$,	$\widetilde{E}(\mu)$:	$\mathcal{O}\left(N^2 \widetilde{N}^2\right)$,
$R\left(\widehat{\psi}(\mu);\mu\right)$:	$\mathcal{O}\left(M_b N^2 \widetilde{N}\right)$,	$E\left(\widehat{\psi}(\mu);\mu\right)$:	$\mathcal{O}\left(N^2 \widetilde{N}\right)$,
$\left\|\widehat{\psi}(\mu)\right\|_X$:	$\mathcal{O}\left(\widetilde{N}^2\right)$.		

Estimation of $\beta(\widehat{u}(\mu), \widehat{h}(\mu))$:

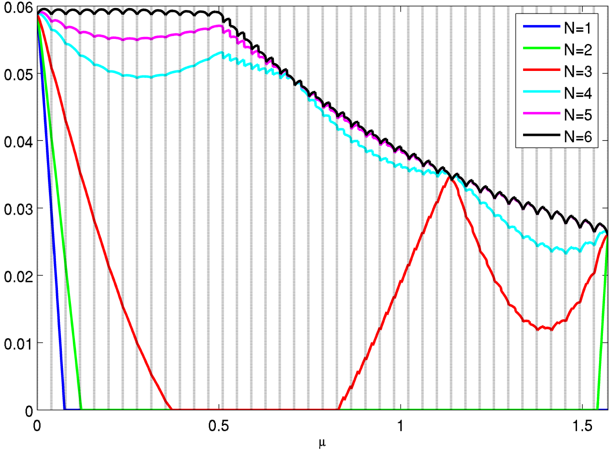
Let $\mathcal{F}(t;\bar{\mu})$ be an expansion of $\left(\beta(\widehat{u}(\mu), \widehat{h}(\mu))\right)^2$ in $\bar{\mu}$, s.t.

Corollary

1. $\mathcal{F}(t;\bar{\mu})$ *is concave in* t, *i.e.* $\forall t \in [t_1, t_2]$: $\mathcal{F}(t;\bar{\mu}) \geq \min\{\mathcal{F}(t_1;\bar{\mu}), \mathcal{F}(t_2;\bar{\mu})\}$.
2. *For given* $\mu, \bar{\mu} \in \mathcal{D}$, $t = \mu - \bar{\mu}$, *the* inf-sup *parameter satisfies*
$$\beta(\widehat{u}(\mu); \widehat{h}(\mu)) \quad \geq \quad \left(\sqrt{\mathcal{F}(t;\bar{\mu})} - \delta(t;\bar{\mu})\right)^+ ,$$

where $\delta(t;\bar{\mu})$ is a **second order correction**.

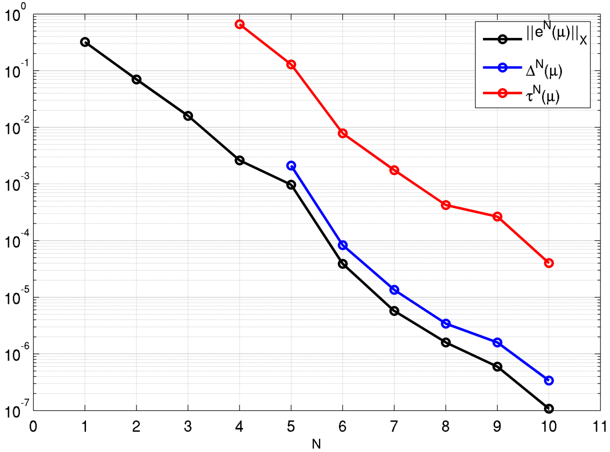
Lower bound for **inf-sup parameter**:



For $a = 0.1$, $\underline{b} = (0.5, 0.5)^T$ and $c = 0$ for the **Primal Problem** we obtain:

N	$\left\|e^N(\mu)\right\|_X$	$\Delta^N(\mu)$	$\eta^N(\mu)$	$\tau^N(\mu)$
2	7.02e-02	NaN	NaN	Inf
4	2.60e-03	NaN	NaN	6.61e-01
6	3.91e-05	8.32e-05	2.34e+00	7.85e-03
8	1.60e-06	3.43e-06	2.04e+00	4.25e-04
10	1.08e-07	3.40e-07	5.03e+00	4.02e-05

Where $e^N(\mu) := u(\mu) - \widehat{u}^N(\mu)$ and $\eta^N(\mu) := \Delta^N(\mu) / \left\|e^N(\mu)\right\|_X$.

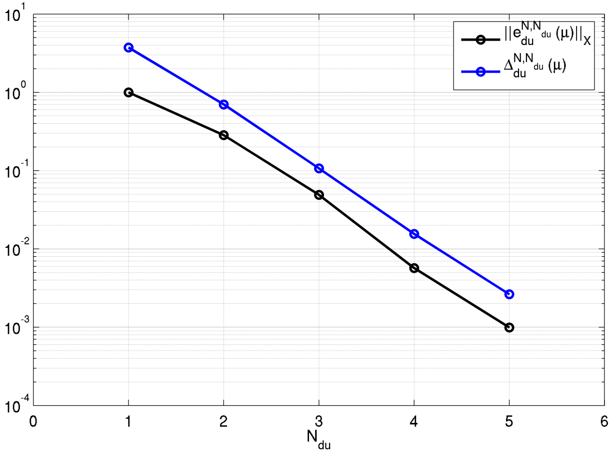


||e^N(μ)||_X
Δ^N(μ)
τ^N(μ)
10^0
10^-1
10^-2
10^-3
10^-4
10^-5
10^-6
10^-7
0
1
2
3
4
5
6
7
8
9
10
11
N

For the **Dual Problem** for fixed $N = N^{\max} = 10$:

$\widetilde{N}$	$\left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X$	$\widetilde{\Delta}^{N,\widetilde{N}}(\mu)$	$\eta^{N,\widetilde{N}}(\mu)$
1	9.99e-01	3.73e+00	3.03e+00
2	2.84e-01	7.00e-01	2.59e+00
3	4.91e-02	1.07e-01	2.52e+00
4	5.69e-03	1.56e-02	2.58e+00
5	9.91e-04	2.64e-03	4.06e+00

Where $\widetilde{e}^{N,\widetilde{N}}(\mu) := \psi^N(\mu) - \widehat{\psi}^{N,\widetilde{N}}(\mu)$ and $\widehat{\eta}^{N,\widetilde{N}}(\mu) := \widetilde{\Delta}^{N,\widetilde{N}}(\mu) / \left\|\widetilde{e}^{N,\widetilde{N}}(\mu)\right\|_X$.



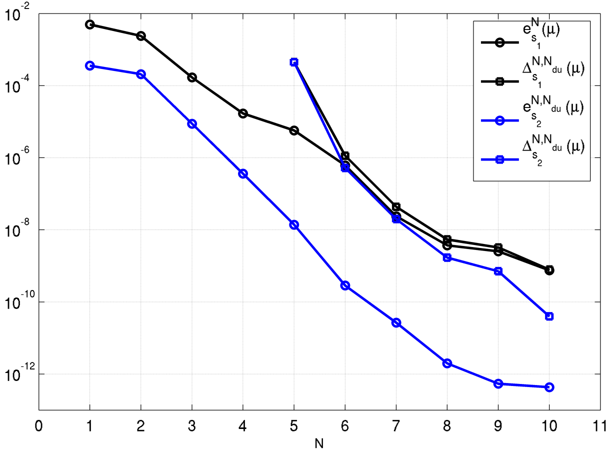
$||e_{du}^{N,N_{du}}(\mu)||_X$
$\Delta_{du}^{N,N_{du}}(\mu)$
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-4}
0
1
2
3
4
5
6
N_{du}

Output Approximation for $\widetilde{N} = \widetilde{N}^{\max} = 5$:

N	$e_{s_1}^{N}(\mu)$	$\Delta_{s_1}^{N,\widetilde{N}}(\mu)$	$\eta_{s_1}^{N,\widetilde{N}}(\mu)$	$e_{s_2}^{N,\widetilde{N}}(\mu)$	$\Delta_{s_2}^{N,\widetilde{N}}(\mu)$	$\eta_{s_2}^{N,\widetilde{N}}(\mu)$
2	2.40e-03	NaN	NaN	2.08e-04	NaN	NaN
4	1.69e-05	NaN	NaN	3.60e-07	NaN	NaN
6	6.17e-07	1.14e-06	1.92e+00	2.86e-10	5.19e-07	2.45e+03
8	3.71e-09	5.40e-09	1.65e+00	1.97e-12	1.69e-09	8.56e+02
10	7.50e-10	7.89e-10	1.06e+00	4.33e-13	3.96e-11	1.66e+02

Where $e_{s_1}^{N}(\mu) := |s(\mu) - s_1^{N}(\mu)|$, $\eta_{s_1}^{N,\widetilde{N}}(\mu) := \Delta_{s_1}^{N,\widetilde{N}}(\mu)/e_{s_1}^{N}(\mu)$,

as well as, $e_{s_2}^{N,\widetilde{N}}(\mu) := |s(\mu) - s_2^{N,\widetilde{N}}(\mu)|$, $\eta_{s_2}^{N,\widetilde{N}}(\mu) := \Delta_{s_2}^{N,\widetilde{N}}(\mu)/e_{s_2}^{N,\widetilde{N}}(\mu)$.


$e_{s_1}^{N}(\mu)$
$\Delta_{s_1}^{N,N_{du}}(\mu)$
$e_{s_2}^{N,N_{du}}(\mu)$
$\Delta_{s_2}^{N,N_{du}}(\mu)$
10^{-2}
10^{-4}
10^{-6}
10^{-8}
10^{-10}
10^{-12}
0 1 2 3 4 5 6 7 8 9 10 11
N

Computational savings (for the output approximation):

	$N = 2$	$N = 4$	$N = 6$	$N = 8$	$N = 10$
$\widetilde{N} = 0$	385	364	346	328	311
$\widetilde{N} = 1$	351	287	261	244	228
$\widetilde{N} = 2$	349	285	258	240	224
$\widetilde{N} = 3$	349	284	257	240	224
$\widetilde{N} = 4$	349	284	257	240	223
$\widetilde{N} = 5$	348	284	257	239	223

All computations are done with *Matlab 6.5* in conjunction with *Femlab 2.3* on an *AMD Opteron Processor 252* at 2.6 GHz

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