Multi scale shape optimization under uncertainty

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Outline

Conceptual sketch

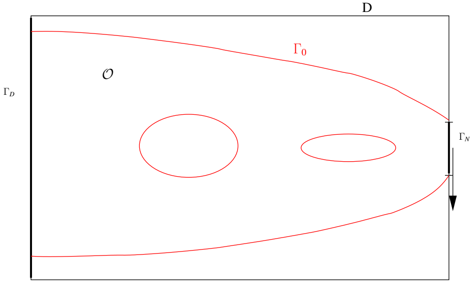

Figure: General setting in 2D

Conceptual sketch

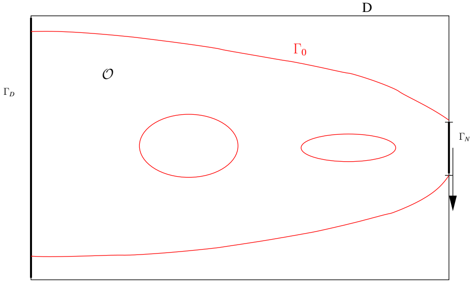

Figure: General setting in 2D

Information Constraint

decide $\mathcal{O}$ $\longrightarrow$ observe $f(\omega), g(\omega)$ $\longrightarrow$ decide $u = u(\mathcal{O}, \omega)$

Linear elasticity model

The displacement u is given by the equation system

PDE

$$\begin{cases} -\mathrm{div}(Ae(u)) = f & \text{in } \mathcal{O}, \\ u = 0 & \text{on } \Gamma_D, \\ (Ae(u))n = g & \text{on } \Gamma_N \end{cases}$$

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- Elastic body $\mathcal{O} \subset \mathbb{R}^3$

$$\partial\mathcal{O} = \Gamma_N \cup \Gamma_D, \ \Gamma_D \neq \emptyset$$

- Volume forces f in $\mathcal{O}$
- Neumann forces g on Γ_N

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where $e(u) = \frac{1}{2}(\nabla u + \nabla u^T)$ is the linearized strain tensor

and Hooke's law

$$A\xi = 2\mu\xi + \lambda(\mathrm{tr}\xi)\mathrm{Id}, \ \text{for any symmetric matrix } \xi$$

Shape optimization problem

- Compliance

$$J(\mathcal{O}) = \int_{\mathcal{O}} f \cdot u \, \mathrm{d}x + \int_{\Gamma_N} g \cdot u \, \mathrm{d}s$$

- Least square error compared to target displacement

$$J(\mathcal{O}) = \left(\int_{\mathcal{O}} |u - u_0|^2 \, \mathrm{d}x \right)^{\frac{1}{2}}$$

Shape optimization problem

$$\min_{\mathcal{O} \in \mathcal{O}_{ad}} \quad J(\mathcal{O}) \; + \; lV(\mathcal{O}) \qquad \text{with } l \in \mathbb{R}, \; l > 0$$

$$\mathcal{O}_{ad} = \{\mathcal{O} \subset D : \; \partial\mathcal{O} \text{ Lipschitz-continuous } \}$$

Outline

Two-Stage Stochastic Linear Program

$$\min\{c^T x + q^T y \ : \ Tx + Wy = z(\omega), \ y \in Y, \ x \in X\}$$

Information Constraint

$$\text{decide } x \ \longrightarrow \ \text{observe } z(\omega) \ \longrightarrow \ \text{decide } y \ = \ y(x, z(\omega))$$

$$\begin{aligned} &\min_x\{c^T x + \min_y\{q^T y : \ Wy = z(\omega) - Tx, \ y \in Y\} && : \ x \in X\} \\ &\min_x\{c^T x + G(x,\omega) && : \ x \in X\} \end{aligned}$$

$\longrightarrow$ looking for a minimal member in family of random variables

$$\{c^T x + G(x,\omega) \ : \ x \in X\}$$

Outline

General Objective Function

$$J(\mathcal{O}, u(\mathcal{O}, \omega)) = \int_{\mathcal{O}} j(u)\,\mathrm{d}x + \int_{\partial\mathcal{O}} k(u)\,\mathrm{d}s + \ell \int_{\mathcal{O}} \mathrm{d}x, \;\; \mathcal{O} \in \mathcal{U}_{\mathrm{ad}}, \ell > 0$$

- $u = u(\mathcal{O}, \omega)$ is the solution of the PDE
- assume $j(.)$ and $k(.)$ are linear or quadratic and independent of ω

The two Stages

We now introduce random forces $f(\omega)$ and $g(\omega)$ to the shape optimization problem.

- **First stage** Non-anticipative decision on $\mathcal{O}$ has to be taken
- observe the random forces $f(\omega)$ and $g(\omega)$ by choosing a scenario
- **Second Stage** The variational formulation of elasticity, given $\mathcal{O}$ and ω, takes the role of the second-stage problem

Information constraint here

decide $\mathcal{O}$ $\longrightarrow$ observe $f(\omega), g(\omega)$ $\longrightarrow$ decide $u = u(\mathcal{O}, \omega)$

Variational two stage formulation

variational description of linearized elasticity:

$$
\begin{aligned}
E(\mathcal{O}, u, \omega) &:= \frac{1}{2} A(\mathcal{O}, u, u) - l(\mathcal{O}, u, \omega) \quad \text{with} \\
A(\mathcal{O}, \psi, \vartheta) &:= \int_{\mathcal{O}} A_{ijkl} e_{ij}(\psi) e_{kl}(\vartheta) dx \\
l(\mathcal{O}, \vartheta, \omega) &:= \int_{\mathcal{O}} f_i(\omega) \vartheta_i dx + \int_{\partial \mathcal{O}} g_i(\omega) \vartheta_i \; d\mathcal{H}^{d-1}
\end{aligned}
$$

Two stage shape optimization problem

$$
\min_{\mathcal{O} \in \mathcal{O}_{ad}} \{ J(\mathcal{O}, \omega) : u(\mathcal{O}, \omega) = argmin_u \, E(\mathcal{O}, u, \omega) \}
$$

$$
\mathcal{O}_{ad} = \{ \mathcal{O} \subset D : \; \partial \mathcal{O} \text{ Lipschitz-continuous} \}
$$

Direct comparison with the linear case

shape optimization : $\min\{J(\mathcal{O},\omega) : u(\mathcal{O},\omega) = \operatorname{argmin}_u E(\mathcal{O},u,\omega)\}$

linear program : $\min\{j(x,\omega) = c^T x + \min\{q^T y \ : \ Wy = z(\omega) - T(x)\}\}$

correspondences:

$$\begin{array}{lcl} \mathcal{O} & \leftrightarrow & x \\ u(\mathcal{O},\omega) & \leftrightarrow & y \\ j(x,\omega) & \leftrightarrow & J(\mathcal{O},\omega) \end{array}$$

optimization task

$$\min\{\mathbb{Q}_{\mathbb{E}}(O) := \mathbb{E}_\omega(J(O,\omega)) : O \in \mathcal{O}_{ad}\}$$

load configuration:

assume that ω follows a discrete distribution with scenarios ω_σ and probabilities π_σ with $\sum_{\sigma=1}^S \pi_\sigma = 1$ and 'basis' loads $(f^k,\ g^m)$ spanning the load space:

$$f(\omega) = \sum_{k=1}^K \alpha_k f^k\,, \qquad g(\omega) = \sum_{m=1}^M \beta_m f^m$$

by linearity :

$$\bar{u}(\mathcal{O},\omega) = \sum_{k=1}^K \alpha_k\, u_f^k + \sum_{m=1}^M \beta_m u_g^m$$

solves $\quad A(\mathcal{O},\bar{u}(\mathcal{O},\omega_\sigma),\varphi) = l(\mathcal{O},\varphi,\omega_\sigma)$

Derivative of the stochcastic functional

Given $\bar{u}(\mathcal{O}, \omega_\sigma)$ we rewrite the stochastic program

$$\min\left\{\mathbb{Q}_{\mathbb{E}}(\mathcal{O}) = \ell \int_{\mathcal{O}} dx \; + \sum_{\sigma=1}^{S} \pi_\sigma \; \int_{\mathcal{O}} j(u)\,\mathrm{d}x + \int_{\partial\mathcal{O}} k(u)\,\mathrm{d}s \quad : \mathcal{O} \in \mathcal{O}_{ad}\right\}$$

we obtain the shape derivative

$$\mathbb{Q}'_{\mathbb{E}}(\mathcal{O})(V) \; = \sum_{\sigma=1}^{S} \pi_\sigma \, J'(\mathcal{O}, \omega_\sigma)(V)$$

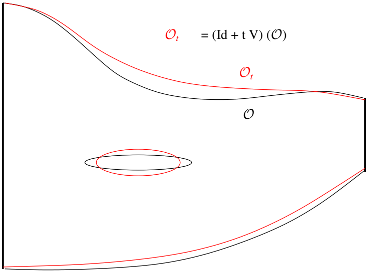
The approach is computationally efficient if $K + M \ll S$.

Outline

Shape gradient

We consider variations $\mathcal{O}_t = (Id + t \cdot V)(\mathcal{O})$, $t > 0$ of a smooth elastic domain $\mathcal{O}$ for a smooth vector field V defined on the working domain D.
The shape derivative of $J(\mathcal{O})$ at $\mathcal{O}$ in direction V is defined as the Fréchet derivative of the mapping t $\rightarrow J(\mathcal{O}_t)$, i.e.

$$J(\mathcal{O}_t) = J(\mathcal{O}) + \left\langle \frac{\partial J}{\partial \mathcal{O}}, V \right\rangle + o(\|V\|)$$



cf. [Sokolowski, Zolesio '92], [Delfour, Zolesio '01]

Shape gradient

As a classical result of the shape sensitivity analysis the shape derivative takes the form

$$< \frac{\partial J}{\partial \mathcal{O}}, V > \; = \int\limits_{\Gamma_N} \left(2 \left[\frac{\partial (g \cdot u)}{\partial n} \; + \; hg \cdot u \; + \; f \cdot u \right] \; - \; \mathcal{A}\epsilon(u) : \epsilon(u) \right) \; V \cdot \vec{n} \, d\nu$$

$$+ \int\limits_{\Gamma_D} \left(\mathcal{A}\epsilon(u) : \epsilon(u) \right) \; V \cdot \vec{n} \, d\nu$$

Here h denotes the mean curvature of $\partial \mathcal{O}$ and $\vec{n}$ the outer normal.

Shape gradient in level set formulation

When the domain $\mathcal{O}$ is implicitly deformed by varying the level set function ϕ

$$\phi_t \; = \; \phi \; + \; t\psi$$

the level set equation

$$\partial_t \phi + |\nabla \phi| \, v \cdot n \; = \; 0 \qquad n = \frac{\nabla \phi}{|\nabla \phi|}$$

allows to define

$$< \frac{\partial J}{\partial \phi}, \psi >:=< \frac{\partial J}{\partial \mathcal{O}}, -\psi \cdot \frac{\vec{n}}{\|\nabla \, \phi\|} >$$

cf. [Osher, Sethian ’88],

[Burger, Osher ’04]

Shape gradient in level set formulation

We take into account a regularized gradient descent, based on the metric

$$\mathcal{G}(\theta, \zeta) = \int_D \theta\zeta + \frac{\sigma^2}{2} \nabla\theta \cdot \nabla\zeta \, \mathrm{d}x$$

which is related to a Gaussian filter with width σ.
The shape gradient is the solution of equation

$$\mathcal{G}(grad_\phi J, \theta) = < \frac{\partial J}{\partial \mathcal{O}}, \theta > \qquad \forall \theta \in H_0^{1,2}(\mathcal{D})$$

Outline

Optimization algorithm

time continuous regularized gradient descent:

$$\partial\phi(t) = -grad_{\phi}J(\phi)$$

with time discrete relaxation :

$$\mathcal{G}(\phi^{k+1} - \phi^{k}, \theta) = -\tau \ < \frac{\partial J}{\partial \mathcal{O}}, \theta > \qquad \forall \theta \in H_0^{1,2}(\mathcal{D})$$

additional ingrediens of the algorithm :

- multigrid method for the primal and the dual problem $(d = 3)$
- preconditioned CG $(d = 2)$
- cascadic optimization (from coarse to fine grid resolution)
- morphological smoothing when switching the grid resolution $(\sigma = 2.5h \text{ or } 4.5h)$

Outline

Test Setting

$\partial\mathcal{O}$ is divided into 3 parts:

- Γ_D: the fixed Dirichlet boundary
- Γ_N: part of the Neumann boundary where the surface loads act on; this is also fixed and does not move during the optimization process
- Γ_0: all other parts of the boundary; this is the only part of $\partial\mathcal{O}$ to be optimized

The objective function (compliance with $f \equiv 0$):

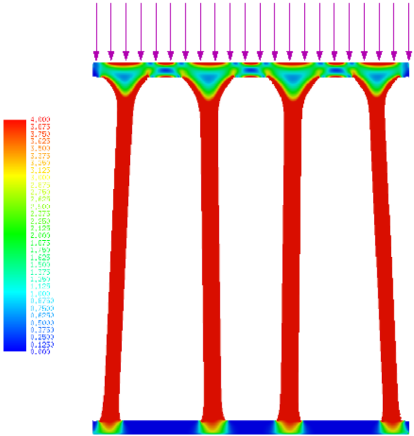
$$J(\mathcal{O},\omega) = \int_{\Gamma_N} g(\omega)\cdot u \,\mathrm{d}s + \ell R_i(\mathcal{O})$$

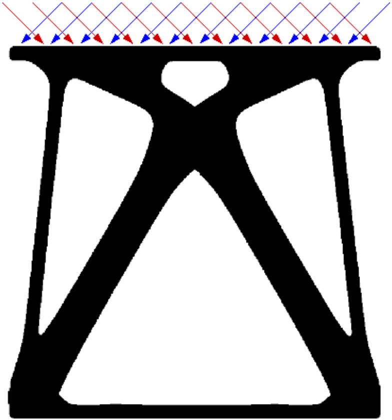
with regularization terms

$$R_1(\mathcal{O}) \quad = \quad \int_{\partial\mathcal{O}} \mathrm{d}s \text{ (and volume preservation)} \, ,$$

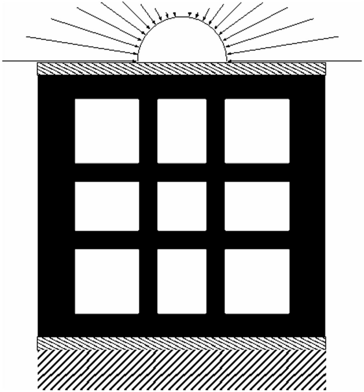
Outline

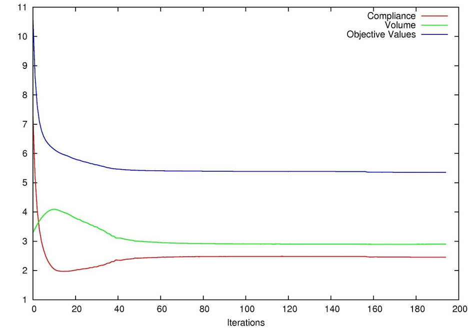
2-Stage vs. Expected Load



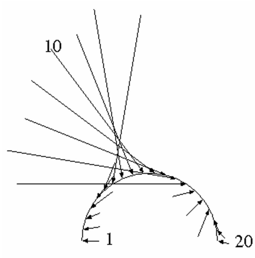


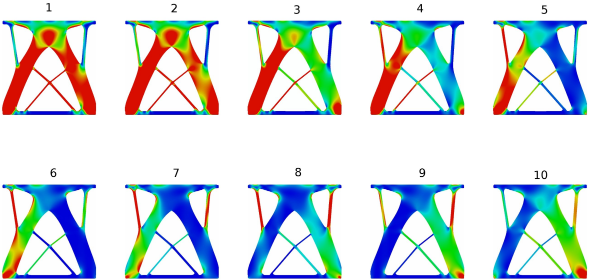
Initial Shape and Objective Values



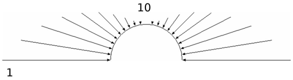


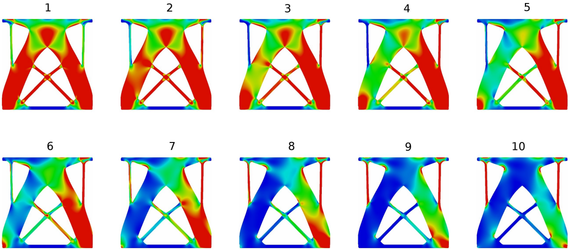
Instance with 20 Scenarios





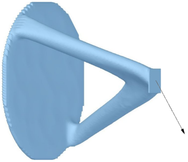
Instance with 21 Scenarios

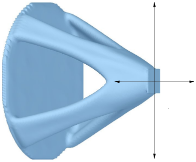


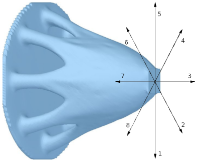


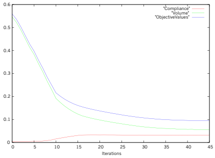
Cantilever

optimal cantilever construction with varying number of scenarios

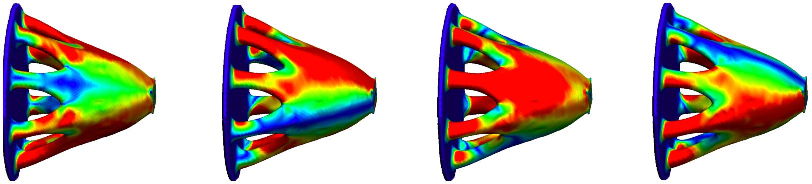






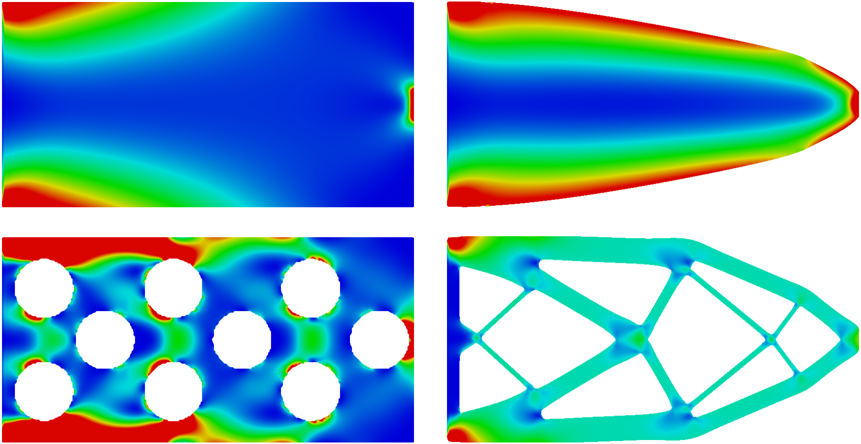


van Mises stress distribution for different scenarios



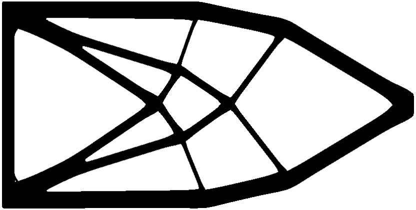
Outline

Different initial shapes yield different solutions



Topological Derivative

$$\mathcal{T}(x) = \lim_{\rho \downarrow 0} \frac{J(\mathcal{O} \setminus \overline{B_\rho(x)}) - J(\mathcal{O})}{|\overline{B_\rho(x)}|}$$



Thank you !