On the Numerical Solution of Differential Operator Riccati Equations in PDE Control

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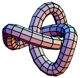

joint work with
Hermann Mena
Escuela Politécnica Nacional, Quito, Ecuador

PDE Constrained Optimization:
Recent Challenges and Future Developments
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Overview

Optimal control of parabolic PDEs

Let us consider optimal control problems for parabolic PDEs

abstract Cauchy problem

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t), \\ \mathbf{x}(0) &= \mathbf{x}_0 \in \mathcal{H} = L_2(\Omega). \end{aligned} \quad (1)$$

output equation

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) \quad (2)$$

LQR problem for the abstract Cauchy equation

Minimize the quadratic cost functional

$$J(\mathbf{u}) = \int_0^{T_f} < \mathbf{y}, \mathbf{Q}\mathbf{y} >_{\mathcal{Y}} + < \mathbf{u}, \mathbf{R}\mathbf{u} >_{\mathcal{U}} dt + < \mathbf{x}_{T_f}, \mathbf{G}\mathbf{x}_{T_f} >_{\mathcal{X}},$$

with respect to the linear constraints (1), (2), $T_f < \infty$.

Optimal control of parabolic PDEs

[Lions '71, Gibson '78, Balakrishnan '77, Lasiecka/Triggiani '00,...]
show that under suitable conditions on $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{Q}, \mathbf{G}$ and $\mathbf{R}$, the optimal control $\mathbf{u}$ is given as the

feedback law

$$\mathbf{u} = -\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}(t)\mathbf{x}(t),$$

where $\mathbf{X}(t)$ is the unique nonnegative solution of the

differential operator Riccati equation

$$\begin{aligned} \dot{\mathbf{X}}(t) &= -\mathfrak{F}(\mathbf{X}(t)) \\ &= -(\mathbf{A}^*\mathbf{X}(t) + \mathbf{X}(t)\mathbf{A} - \mathbf{X}(t)\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^*\mathbf{X}(t) + \mathbf{C}^*\mathbf{Q}\mathbf{C}), \\ \mathbf{X}(T_f) &= \mathbf{G}. \end{aligned}$$

Note: $\dot{\mathbf{X}}(t) = -\mathfrak{F}(\mathbf{X}(t)) \;\Leftrightarrow\; \frac{d}{dt} < \mathbf{v}, \mathbf{X}(t)\mathbf{w} >= - < \mathbf{v}, \mathfrak{F}(\mathbf{X}(t))\mathbf{w} > \quad \forall \mathbf{v}, \mathbf{w} \in \mathrm{dom}(\mathbf{A})$

Optimal control of parabolic PDEs

Consider a semi-discrete LQR problem for a parabolic PDE on $\mathcal{H}^N$

$$\begin{aligned} \dot{x}^N &= A^N x^N + B^N u, \\ y^N &= C^N x^N \end{aligned}$$

with cost function

$$J_N(u) = \int_0^{T_f} < y^N, Q^N y^N > + < u, Ru > dt + < x_{T_f}, G_N x_{T_f} >,$$

then u^N is given in feedback form as

$$u^N = -R^{-1}(B^N)^T X^N x^N,$$

where X_N is the solution of the

Differential Riccati Equation (DRE)

$$\begin{aligned} \dot{X}^N &= -(C^N Q^N C^N + (A^N)^T X^N + X^N A^N \\ &\quad -X^N B^N R^{-1} (B^N)^T X^N), \\ X^N(T_f) &= G^N. \end{aligned}$$

Optimal control of parabolic PDEs

Consider a semi-discrete LQR problem for a parabolic PDE on $\mathcal{H}^N$

$$\begin{aligned} M\dot{x}^N &= A^N x^N + B^N u, \\ y^N &= C^N x^N \end{aligned}$$

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convergence results for approximate DRE solution operators.

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Previous results:

- Convergence in terms of Riccati integral equations [Gibson 1979].
- Approximation schemes and convergence rates [Ito 1991, Kroller/Kunisch 1991, Lasiecka/Triggiani 2000, ...], mostly
  - in terms of Riccati integral equations,
  - for distributed control,
  - for autonomous systems (except for [Kroller/Kunisch 1991]),
  - assuming $\mathcal{H}^N \subset \mathcal{H}$ for approximating spaces.

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convergence results for approximate DRE solution operators.

Here: convergence results

- for differential Riccati equations,
- for distributed and (some) boundary control problems,
- for autonomous and non-autonomous systems,
- for $\mathcal{H}^N \subset \mathcal{H}$ and $\mathcal{H}^N \not\subset \mathcal{H}$ for approximating spaces,

but no convergence rates.

Note: some of our results may be corollaries of [Kroller/Kunisch 1991].

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General assumptions:

- $(\mathcal{H}, ||.||)$, $(\mathcal{H}^N, ||.||_N)$ are Hilbert spaces, in general $\mathcal{H}^N \nsubseteq \mathcal{H}$.
- $\mathbf{A}$, A^N generate strongly continuous semigroups $\mathbf{T}$, T^N on $\mathcal{H}, \mathcal{H}^N$.
- $P^N : \mathcal{H} \to \mathcal{H}^N$, $||P^N \phi||_N \to ||\phi||$ for all $\phi \in \mathcal{H}$.

  (P^N is the canonical orthogonal projection.)
- $B^N \in \mathcal{L}(\mathcal{U}, \mathcal{H}^N)$, G^N, $Q^N \in \mathcal{L}(\mathcal{H}^N)$, $Q^N, G^N \geq 0$.
- For simplicity we will not consider an output equation.

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Autonomous case

Let $T^N(t)$ be a sequence of strongly continuous semigroups on $\mathcal{H}^N$ with infinitesimal generator $A^N \in \mathcal{L}(\mathcal{H}^N)$

Assumptions (H)

Similar to [Banks/Kunisch 1984]:

(i) For all $\varphi \in \mathcal{H}$ it holds that $T^N(t)P^N\varphi \to \mathbf{T}(t)\varphi$ uniformly on any bounded subinterval of $[0, T_f]$.

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(iii) For all $v \in \mathcal{U}$ it holds $B^N v \to \mathbf{B}v$ and for all $\varphi \in \mathcal{H}$ it holds that $B^{N*}P^N\varphi \to \mathbf{B}^*\varphi$.

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Autonomous case

Theorem 1

Let (H) hold, then

$$u^N \to u \quad \text{uniformly on } [0, T_f],$$
$$x^N \to x \quad \text{uniformly on } [0, T_f],$$

and for $\varphi \in \mathcal{H}$,

$$X^N(t)P^N\varphi \to \mathbf{X}(t)\varphi \quad \text{uniformly in } t \in [0, T_f].$$

Here u^N, u, x^N, x denote optimal controls and trajectories of the finite and infinite dimensional problems, respectively.

Outline of Proof.

- Consider a related family of LQR problems defined on $\mathcal{H}^N$.
- Prove that the solution of the corresponding DRE is $X^N(t)P^N$.
- Apply theorems of [Curtain/Pritchard 1978] and [Gibson 1979].

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Assumptions ($\mathcal{H}^N \not\subseteq \mathcal{H}$) (H')

(i) There exist constants M, ω such that $\|T^N(t)\|_N \leq Me^{\omega t}$ for all N and for each $\phi \in \mathcal{H}$, $\|T^N(t)P^N\phi - P^N\mathbf{T}(t)\phi\|_N \to 0$ as $N \to \infty$, uniformly on any bounded subinterval of $[0, T_f]$.

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(vi) For all N, the operators Q^N, G^N are nonnegative self-adjoint.

Autonomous case

Assumptions ($\mathcal{H}^N \not\subseteq \mathcal{H}$) (H')

(i) There exist constants M, ω such that $\|T^N(t)\|_N \leq Me^{\omega t}$ for all N and for each $\phi \in \mathcal{H}, \|T^N(t)P^N\phi - P^N\mathbf{T}(t)\phi\|_N \to 0$ as $N \to \infty$, uniformly on any bounded subinterval of $[0, T_f]$.

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Autonomous case

Theorem 2

Let (H') hold, then

$$u^N \to u \quad \text{uniformly on } [0, T_f],$$
$$x^N \to x \quad \text{uniformly on } [0, T_f],$$

and for $\varphi \in \mathcal{H}$,

$$\|X^N(t)P^N\varphi - P^N\mathbf{X}(t)\varphi\|_N \to 0 \quad \text{uniformly in } t \in [0, T_f].$$

Here u^N, u, x^N, x denote optimal controls and trajectories of the finite and infinite dimensional problems , respectively.

Proof. Corollary of Theorem 4 ($\to$ later).

Non-autonomous case

Let $\mathbf{U}(\cdot,\cdot), U^N(\cdot,\cdot)$ be evolution operators on $\mathcal{H}, \mathcal{H}^N$ with generators $\mathbf{A}(\cdot) \in \mathcal{L}(\mathcal{H}), A^N(\cdot) \in \mathcal{L}(\mathcal{H}^N)$.

Assumptions (NH)

Suppose that, for each $\varphi \in \mathcal{H}$ and $v \in \mathcal{U}$,

(i)	$U^N(t,s)P^N\varphi \to \mathbf{U}(t,s)\varphi$	strongly,	$t_0 \leq s \leq t \leq \mathbf{T}$,
(ii)	$(U^N)^*(t,s)P^N\varphi \to \mathbf{U}^*(t,s)\varphi$	strongly,	$t_0 \leq s \leq t \leq \mathbf{T}$,
(iii)	$B^N(t)v \to \mathbf{B}(t)v$	strongly a.e.,	
(iv)	$B^{N*}(t)P^N\varphi \to \mathbf{B}^*(t)\varphi$	strongly a.e.,	
(v)	$Q^N(t)P^N\varphi \to \mathbf{Q}(t)\varphi$	strongly a.e.,	
(vi)	$G^NP^N\varphi \to \mathbf{G}\varphi$	strongly,	

for $N \to \infty$.

Let $\|U^N(t,s)\|$, $\|B^N\|$, $\|Q^N\|$, $\|\mathbf{R}^N\|$, $\|G^N\|$ be uniformly bounded in N, t, and s and require a constant m such that for each N, $\mathbf{R}(t) \geq m > 0$ for almost all t.

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(v)	$Q^N(t)P^N\varphi \to \mathbf{Q}(t)\varphi$	strongly a.e.,	
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Non-autonomous case

Theorem 3

Let (NH) hold, then

$$\begin{aligned} u^N(t) &\to u(t) \quad \text{strongly a.e. and in } L^2(0, T_f; \mathcal{U}), \\ x^N(t) &\to x(t) \quad \text{strongly pointwise and in } L^2(0, T_f; \mathcal{H}), \end{aligned} \tag{3}$$

and for $\varphi \in \mathcal{H}$,

$$X^N(t)P^N\varphi \to \mathbf{X}(t)\varphi \quad \text{strongly pointwise and in } L^2(0, T_f; \mathcal{H}). \tag{4}$$

If $\mathbf{U}(\cdot, \cdot)$ is strongly continuous and $\mathbf{B}(\cdot)$, $\mathbf{B}^*(\cdot)$, $\mathbf{Q}(\cdot)$, and $\mathbf{R}(\cdot)$ are piecewise strongly continuous, uniform convergence in (NH) implies uniform convergence in (3)–(4).

Outline of Proof.

- Analogous to Theorem 1, consider a related family of LQR problems defined on $\mathcal{H}^N$.
- Prove that the solution of the corresponding DRE is $X^N(t)P^N$.

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Assumptions ($\mathcal{H}^N \not\subseteq \mathcal{H}$) (NH')

(i) There exist M, ω such that $\|U^N(t,s)\|_N \leq Me^{\omega(t-s)}$, $t \geq s$, for all N and for each $\phi \in \mathcal{H}$, $\|U^N(t,s)P^N\phi - P^N\mathbf{U}(t,s)\phi\|_N \to 0$ as $N \to \infty$, uniformly on any bounded subinterval of $[0, T_f]$.

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Theorem 4

Let (NH') hold, then

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and for $\varphi \in \mathcal{H}$,

$$X^N(t)P^N\varphi \to \mathbf{X}(\mathbf{t})\varphi \quad \text{uniformly on } [0, T_f].$$

Here u^N, u, x^N, x denote the optimal control and trajectories for the finite and infinite dimensional problems, respectively.

Outline of Proof. Follows mainly as a consequence of the repeated application of a general convergence result for non-autonomous operators.

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Summarizing theorem

Let us consider a sequence of control problems related to $J_N(u)$.

Theorem

Under suitable conditions on $\mathbf{A}, \mathbf{B}, \mathbf{Q}, \mathbf{G}$ and A^N, B^N, Q^N, G^N we have

$$u^N \to u, \quad x^N \to x \quad \text{uniformly on } [0, T_f],$$

and for $\varphi \in \mathcal{H}$,

$$X^N(t)P^N\varphi \to \mathbf{X}(t)\varphi \quad \text{uniformly in } t \in [0, T_f].$$

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Numerical methods for DREs

- Solution of large-scale DREs by ordinary ODE solvers possible: unrolling matrices into vector $\rightsquigarrow$ vector ODE in n^2 (or $\frac{1}{2}n(n+1)$ if symmetry is exploited) unknowns.
  $\Rightarrow$ Computationally infeasible for 2D/3D problems.
- Our approach (following earlier work by Choi/Laub, Dieci,...): Derive matrix versions of suitable ODE solvers feasible for large-scale computations, exploiting sparsity and low-rank structure of coefficients:

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  Note: due to stiffness, need implicit methods.

Numerical methods for DREs

- Solution of large-scale DREs by ordinary ODE solvers possible: unrolling matrices into vector $\rightsquigarrow$ vector ODE in n^2 (or $\frac{1}{2}n(n+1)$ if symmetry is exploited) unknowns.
  $\Rightarrow$ Computationally infeasible for 2D/3D problems.
- Our approach (following earlier work by Choi/Laub, Dieci,...): Derive matrix versions of suitable ODE solvers feasible for large-scale computations, exploiting sparsity and low-rank structure of coefficients:
- BDF methods [B./Mena 2004].
  - require solution of one ARE/time step,
  - use Newton-ADI with X_k^N as initial guess,
  - main technical difficulty: step size and order control using factors of the solutions only.
  - Variable order code uses orders 1–3.

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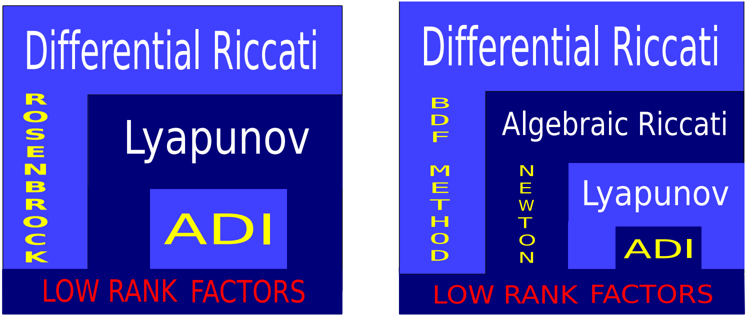
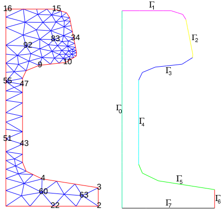
- Our approach (following earlier work by Choi/Laub, Dieci,...): Derive matrix versions of suitable ODE solvers feasible for large-scale computations, exploiting sparsity and low-rank structure of coefficients:
- Rosenbrock methods [B./Mena 2007].
  - require solution of one Lyapunov equation/stage (Lyapunov equations for different stages share the same Lyapunov operator!),
  - use low-rank ADI for Lyapunov equations,
  - main technical difficulty: step size control using factors of the solutions only.
  - Very efficient: Steihaug/Wolfbrand method of 2nd order, variable order code uses orders 1-2.

Matrix versions of the ODE methods



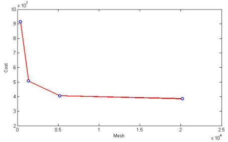
Example 1

- Mathematical model: boundary control for linearized 2D heat equation.



$$
\begin{aligned}
c \cdot \rho \frac{\partial}{\partial t} x &= \lambda \Delta x, && \xi \in \Omega, \\
\lambda \frac{\partial}{\partial n} x &= \kappa (u_k - x), && \xi \in \Gamma_k, \; k = 1:7, \\
\frac{\partial}{\partial n} x &= 0, && \xi \in \Gamma_0.
\end{aligned}
$$

- FEM discretization, different models for initial mesh ($n = 371$), 1, 2, 3, steps of mesh refinement $\Rightarrow$ $n = 1357, 5177, 20209$.



Convergence ($T_f = \infty$): [B./Saak 2005].

Math. model: [Tröltzsch/Unger 1999/2001], [Penzl 1999] and [Saak 2003].

Example 2

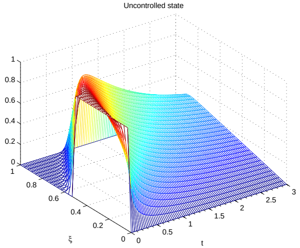
We consider the Burgers equation

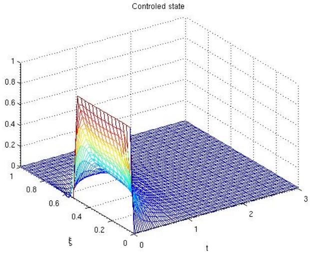
$$
\begin{aligned}
x_t(t,\xi) &= \nu x_{\xi\xi}(t,\xi) - x(t,\xi)x_\xi(t,\xi) \\
&\quad + B(\xi)u(t) + F(\xi)v(t), \\
x(t,0) &= x(t,1) = 0, \qquad t > 0, \\
x(0,\xi) &= x_0(\xi) + \eta_0(\xi), \quad \xi \in]0,1[
\end{aligned}
$$

and the observation process

$$y(t,\xi) = Cx(t,\xi) + w(t,\xi).$$

- Aim is to control the state to 0.
- Consider disturbances in state, output, initial condition.
- Use LQG design within MPC framework based on DRE and compare to ARE approach [Ito/Kunisch 2001–03].





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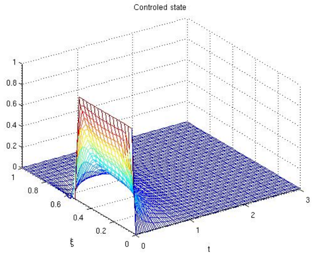
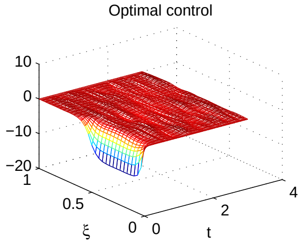
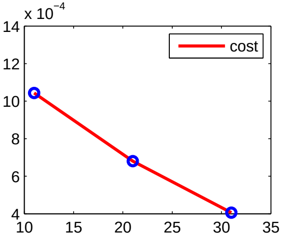
Values of cost function ($*1000$)

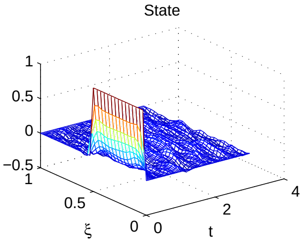
n	ARE	DRE	reduction
without noise in x_0			
30	11.5	9.8	14.8%
201	9.7	8.0	17.5%
with noise in x_0			
31	13.1	11.4	13.0%
201	14.6	12.8	12.3%



Numerical examples
Example 2







Numerical examples

Example 2