An Overview of (Anti-)Localisation Cardinals on Products of Discrete Spaces

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The **Baire space** is the set $\omega^\omega = \{ f \mid f : \omega \to \omega \}$, with clopen sets $[s] = \{ f \in \omega^\omega \mid s \subseteq f \}$ for $s \in <\omega^\omega$ generating the topology. Let $\mathcal{N}$ and $\mathcal{M}$ resp. be the $\sigma$-ideals of Lebesgue null and meagre sets.

\[
\begin{align*}
\text{cov}(\mathcal{I}) &= \min \left\{ |C| \mid C \subseteq \mathcal{I} \text{ and } \bigcup C = \omega^\omega \right\}, \\
\text{non}(\mathcal{I}) &= \min \left\{ |N| \mid N \subseteq \omega^\omega \text{ and } N \notin \mathcal{I} \right\}, \\
\text{add}(\mathcal{I}) &= \min \left\{ |A| \mid A \subseteq \mathcal{I} \text{ and } \bigcup A \notin \mathcal{I} \right\}, \\
\text{cof}(\mathcal{I}) &= \min \left\{ |J| \mid J \subseteq \mathcal{I} \text{ and } \forall X \in \mathcal{I} \exists Y \in J (X \subseteq Y) \right\}.
\end{align*}
\]

\[
\begin{align*}
\text{cov}(\mathcal{N}) &\to \text{non}(\mathcal{M}) \to \text{cof}(\mathcal{M}) \to \text{cof}(\mathcal{N}) \to 2^{\aleph_0} \\
\aleph_1 &\to \text{add}(\mathcal{N}) \to \text{add}(\mathcal{M}) \to \text{cov}(\mathcal{M}) \to \text{non}(\mathcal{N})
\end{align*}
\]
Let \( \kappa \) be uncountable, then \( {}^\kappa \kappa \) is a **generalised Baire space**. We say \( f \in {}^\kappa \kappa \) are \( \kappa \)-reals. If \( \kappa \) is strongly inaccessible, we can generalise the middle part of the Cichoń Diagram:

\[
\begin{array}{c}
\non(\mathcal{M}_\kappa) & \rightarrow & \cof(\mathcal{M}_\kappa) & \rightarrow & 2^\kappa \\
\uparrow & & \uparrow & & \\
\mathcal{b}_\kappa & \rightarrow & \mathcal{d}_\kappa & & \\
\uparrow & & \uparrow & & \\
\kappa^+ & \rightarrow & \add(\mathcal{M}_\kappa) & \rightarrow & \cov(\mathcal{M}_\kappa)
\end{array}
\]

There is no Lebesgue measure on \( {}^\kappa \kappa \), so there is no generalisation of \( \mathcal{N} \) to \( {}^\kappa \kappa \). We can generalise \( \add(\mathcal{N}) \) and \( \cof(\mathcal{N}) \) using a combinatorial definition instead.
• Slaloms & cardinal characteristics
  ○ Relations and consistency results for $\omega$
  ○ Relations for $\kappa$
  ○ Consistency results for $b_{\kappa}^{b,h}(\in^*)$ and $d_{\kappa}^{b,h}(\in^*)$
  ○ Consistency results for $b_{\kappa}^{b,h}(\in^\infty)$ and $d_{\kappa}^{b,h}(\in^\infty)$
Let $\kappa$ be regular strong limit and let $h, b$ be increasing cardinal function with domain $\kappa$. We define the bounded space
\[ \prod b = \prod_{\alpha \in \kappa} b(\alpha) = \{ f : \kappa \rightarrow \text{Ord} \mid \forall \alpha < \kappa (f(\alpha) < b(\alpha)) \}. \]

Let $\varphi$ with $\text{dom}(\varphi) = \kappa$ be an $(h, b)$-slalom if $\varphi(\alpha) \in [b(\alpha)]^{<h(\alpha)}$ for all $\alpha \in \kappa$, and $\text{Loc}^b_h$ be the set of $(h, b)$-slaloms.

For $f, g \in \prod b$ and $\varphi \in \text{Loc}^b_h$, we say
- $f \in^* \varphi$ iff $f(\alpha) \in \varphi(\alpha)$ for almost all $\alpha < \kappa$,
- $f \in^\infty \varphi$ iff $f(\alpha) \in \varphi(\alpha)$ for cofinally many $\alpha < \kappa$,
- $f =^\infty g$ iff $f(\alpha) = g(\alpha)$ for cofinally many $\alpha < \kappa$,
- $f \leq^* g$ iff $f(\alpha) \leq g(\alpha)$ for almost all $\alpha < \kappa$. 
Let $\mathcal{R} = \langle X, Y, R \rangle$ be a relational system, i.e. $R \subseteq X \times Y$. Let $\|\mathcal{R}\| = \min \{|Z| \mid Z \subseteq Y \text{ and } \forall a \in X \exists b \in Z (a R b)\}$ be the norm of $\mathcal{R}$. The dual $\mathcal{R}^{-1} = \{(b, a) \in Y \times X \mid (a, b) \notin R\}$ provides a dual relational system $\mathcal{R}^\perp = \langle Y, X, \mathcal{R}^{-1} \rangle$.

Given $\mathcal{R} = \langle X, Y, R \rangle$ and $\mathcal{R}' = \langle X', Y', R' \rangle$, a Tukey connection from $\mathcal{R}$ to $\mathcal{R}'$ is a pair $\rho_- : X \to X'$ and $\rho_+ : Y' \to Y$ such that for any $x \in X$ and $y' \in Y'$ with $(\rho_-(x), y') \in R'$ we also have $(x, \rho_+(y')) \in R$. We let $\mathcal{R} \preceq \mathcal{R}'$ denote that there exists a Tukey connection from $\mathcal{R}$ to $\mathcal{R}'$, and $\mathcal{R} \equiv \mathcal{R}'$ that $\mathcal{R} \preceq \mathcal{R}' \preceq \mathcal{R}$.

**Lemma**

If $\mathcal{R} \preceq \mathcal{R}'$, then $\|\mathcal{R}\| \leq \|\mathcal{R}'\|$ and $\|\mathcal{R}'^\perp\| \leq \|\mathcal{R}^\perp\|$. 

We define the following relational systems:

\[ \mathcal{L}_h^b = \langle \prod b, \text{Loc}_h^b, \in^* \rangle \quad \text{(localisation)} \]

\[ \mathcal{AL}_h^b = \langle \prod b, \text{Loc}_h^b, \in^\infty \rangle \quad \text{(anti-localisation)} \]

\[ \mathcal{ED}^b = \langle \prod b, \prod b, \neq^\infty \rangle \quad \text{(eventually different / cofinally equal)} \]

\[ \mathcal{D}^b = \langle \prod b, \prod b, \leq^* \rangle \quad \text{(dominating / unbounding)} \]

with the norms:

\[ \| \mathcal{L}_h^b \| = \vartheta^b_{\kappa} (\in^*) \]

\[ \| \mathcal{AL}_h^b \| = \vartheta^b_{\kappa} (\in^\infty) \]

\[ \| \mathcal{ED}^b \| = \vartheta^b_{\kappa} (\neq^\infty) \]

\[ \| \mathcal{D}^b \| = \vartheta^b_{\kappa} (\leq^*) \]
○ Slaloms & localisation cardinals

● Relations and consistency results for $\omega$

○ Relations for $\kappa$

○ Consistency results for $b_{\kappa}^{b,h}(\in^*)$ and $d_{\kappa}^{b,h}(\in^*)$

○ Consistency results for $b_{\kappa}^{b,h}(\in^\infty)$ and $d_{\kappa}^{b,h}(\in^\infty)$
Let \( \overline{\omega} : n \mapsto \omega \) for all \( n \in \omega \) and let \( h \in \omega \omega \) be cofinal.

**Theorem**  Bartoszyński [1987]

\[
\begin{align*}
\overline{b}_{\omega,h}^{(\infty)} & = \mathrm{add}(\mathcal{N}) \quad \text{and} \\
\overline{d}_{\omega,h}^{(\infty)} & = \mathrm{cof}(\mathcal{N}).
\end{align*}
\]

**Theorem**  Bartoszyński [1987] or Bartoszyński and Judah [1995]

\[
\begin{align*}
\overline{d}_{\omega,h}^{(\infty)} & = \overline{b}_{\omega}^{(\not= \infty)} = \mathrm{non}(\mathcal{M}) \quad \text{and} \\
\overline{b}_{\omega,h}^{(\infty)} & = \overline{d}_{\omega}^{(\not= \infty)} = \mathrm{cov}(\mathcal{M}).
\end{align*}
\]

**Proposition**

\[
\begin{align*}
\overline{b}_{\omega,\overline{\omega}}^{(\infty)} & = \overline{d}_{\omega,\overline{\omega}}^{(\infty)} = \overline{b}_{\omega}^{(\leq \infty)} = b \quad \text{and} \\
\overline{d}_{\omega,\overline{\omega}}^{(\infty)} & = \overline{b}_{\omega,\overline{\omega}}^{(\infty)} = \overline{d}_{\omega}^{(\leq \infty)} = d.
\end{align*}
\]
Theorem Cardona et al. [2021]

It is consistent that there exist $h_\xi, b_\xi \in \omega \omega$ for each $\xi < c$ and a strictly increasing sequence of cardinals $\langle \kappa_\xi \mid \xi < c \rangle$ such that $b_{\omega, h}^{b,h}(\in^*) = d_{\omega, h}^{b,h}(\in^*) = b_{\omega, h}^{b,h}(\in^\infty) = d_{\omega, h}^{b,h}(\in^\infty) = \kappa_\xi$.

The proof is the culmination of the investigation into these cardinals using creature forcings, originating from Goldstern and Shelah [1993] and improved by Kellner and Shelah, and later in connection with Yorioka ideals in several papers by Kamo, Osuga, Brendle, Mejía, Klausner and Cardona.
Contents

- Slaloms & localisation cardinals
- Relations and consistency results for $\omega$
  - Relations for $\kappa$
    - Consistency results for $b_{\kappa}^{b,h}(\in^*)$ and $d_{\kappa}^{b,h}(\in^*)$
    - Consistency results for $b_{\kappa}^{b,h}(\in^\infty)$ and $d_{\kappa}^{b,h}(\in^\infty)$
For some choices of $b$ and $h$, the bounded (anti-)localisation cardinals may be trivial.

**Lemma**

\[ \partial^{b,h}_\kappa(\in^*) = 1 \text{ iff } b <^* h, \text{ which implies } b^{b,h}_\kappa(\in^*) \text{ is undefined.} \]

\[ \partial^{b,h}_\kappa(\in^\infty) = 1 \text{ iff } b <^\infty h, \text{ which implies } b^{b,h}_\kappa(\in^\infty) \text{ is undefined.} \]

**Lemma**  Cardona and Mejía [2019] & Goldstern and Shelah [1993] ($\kappa = \omega$)

If $\lambda < \kappa$ exists and is minimal s.t. \( D_\lambda = \{ \alpha \in \kappa \mid h(\alpha) = \lambda \} \) is cofinal in $\kappa$, then $b^{b,h}_\kappa(\in^*) = \lambda$ and $2^\kappa \leq \partial^{b,h}_\kappa(\in^*)$. If no such $\lambda$ exists, $\kappa^+ \leq b^{b,h}_\kappa(\in^*)$, and if also $b \leq 2^\kappa$, then $\partial^{b,h}_\kappa(\in^*) \leq 2^\kappa$. 
Let increasing $f : \kappa \rightarrow \text{Ord}$ be **continuous** at $\gamma \in \kappa$ if
\[ f(\gamma) = \bigcup_{\alpha < \gamma} f(\alpha). \]
We call $f$ stationarily continuous there exists $S$ stationary in $\kappa$ s.t. $f$ is continuous at all limit $\gamma \in S$.

**Lemma**
For $\lambda < \kappa$ let
\[ D_\lambda = \{ \alpha \in \kappa \mid b(\alpha) \leq \lambda \} \cup \{ \alpha \in \kappa \mid h(\alpha) = b(\alpha) \land \text{cf}(b(\alpha)) \leq \lambda \}. \]

- (i) If $\lambda < \kappa$ exists and is minimal s.t. $D_\lambda$ is cofinal in $\kappa$, then $d_{b,h}^\kappa(\in \infty) = \lambda$.
- (ii) If all $D_\lambda$ are bounded, $b$ is stat.cont., then $d_{b,h}^\kappa(\in \infty) = \kappa$.
- (iii) If all $D_\lambda$ are bounded, $b$ is not stat.cont., then $\kappa^+ \leq d_{b,h}^\kappa(\in \infty)$.

A dual result for the relation between $b_{\kappa}^{b,h}(\in \infty)$ and $2^\kappa$ is not known yet.
Regarding $b$ and $h$

If $h \preceq^* h'$ and $b \succeq^* b'$, then:

$$AL^b_{h'} \preceq \preceq \preceq AL^b_h \preceq \preceq L^b_{h'}$$

$$\mathcal{D}^b \preceq \preceq \preceq \mathcal{D}^b_{h'}$$

$$d^b_{\kappa}(\infty) \leq d^b_{\kappa}(\infty) \leq d^b_{\kappa}(\infty) \leq d^b_{\kappa}(\infty)$$

$$b^b_{\kappa}(\infty) \leq b^b_{\kappa}(\infty) \leq b^b_{\kappa}(\infty) \leq b^b_{\kappa}(\infty)$$

$$d^b_{\kappa}(\neq \infty) \leq d^b_{\kappa}(\neq \infty) \leq d^b_{\kappa}(\neq \infty) \leq d^b_{\kappa}(\neq \infty)$$

$$b^b_{\kappa}(\neq \infty) \leq b^b_{\kappa}(\neq \infty) \leq b^b_{\kappa}(\neq \infty) \leq b^b_{\kappa}(\neq \infty)$$
(Anti-)Localisation on $\kappa$)

Let $\bar{\kappa} : \alpha \mapsto \kappa$ for all $\alpha \in \kappa$.

The relation between eventual difference and the meagre ideal generalise to strongly inaccessible $\kappa$.

**Theorem**  
Landver [1992] and Blass et al. [2005]

$$d_{\kappa}^\kappa(\neq \infty) = \text{cov}(\mathcal{M}_\kappa) \text{ and } b_{\kappa}^\kappa(\neq \infty) = \text{non}(\mathcal{M}_\kappa).$$

**Theorem**  
Brendle et al. [2018]

\[
\max \{ \text{non}(\mathcal{M}_\kappa), d_{\kappa}^\kappa(\leq^*) \} = \text{cof}(\mathcal{M}_\kappa) \leq d_{\kappa}^\kappa,h(\in^*), \text{ and } \\
\min \{ \text{cov}(\mathcal{M}_\kappa), b_{\kappa}^\kappa(\leq^*) \} = \text{add}(\mathcal{M}_\kappa) \geq b_{\kappa}^\kappa,h(\in^*).
\]

**Proposition**

If $h \in \kappa$, then $d_{\kappa}^{\kappa,h}(\in^\infty) = b_{\kappa}^\kappa(\neq \infty)$ and $b_{\kappa}^{\kappa,h}(\in^\infty) = d_{\kappa}^\kappa(\neq \infty)$.

We will state a more general result.
Say that \( b \) **overshadows** \( h \) if there exists an interval partition 
\[ \langle I_\alpha \mid \alpha < \kappa \rangle \] of \( \kappa \) with \( |I_\alpha| = h(\alpha) \) for each \( \alpha \in \kappa \) such that 
\[ b(\alpha) = b(\xi) = b(\alpha)^{h(\alpha)} \] for all \( \xi \in I_\alpha \) and \( \alpha \in \kappa \).

**Theorem**

If \( b \) overshadows \( h \), then 
\[ b_{\kappa,h}(\infty) = b_{\kappa}(\neq \infty) \] and 
\[ b_{\kappa,h}(\infty) = d_{\kappa}(\neq \infty). \]

Note that \( \kappa \) overshadows any \( h \in \kappa \cdot \kappa \). In particular, the cardinalities 
of \( d_{\kappa,h}(\infty) \) and \( b_{\kappa,h}(\infty) \) do not depend on the choice of \( h \in \kappa \cdot \kappa \).
Nontrivial cases

Let $h \leq h' \leq \kappa$ be increasing cofinal and $h \in {}^\kappa \kappa$. If $h' =^* b$, the dotted lines are equality.
Nontrivial cases

Let $h \leq b' \leq b \in \kappa$ be increasing cofinal and $b$ overshadows $h$.

\[
\begin{align*}
&\kappa^b(\leq^*) \rightarrow \kappa^b(\neq \infty) = \kappa^{b',h}(\in \infty) \rightarrow \kappa^{b',h}(\in^*) \\
\uparrow &
\end{align*}
\]

\[
\begin{align*}
&\kappa^{b'}(\leq^*) \rightarrow \kappa^{b'}(\neq \infty) \\
\uparrow &
\end{align*}
\]

\[
\begin{align*}
&\kappa^{b',h}(\in \infty) \rightarrow \kappa^{b',h}(\in^*) \\
\uparrow &
\end{align*}
\]

\[
\begin{align*}
&\kappa^{b',h}(\in^*) \rightarrow \kappa^{b',h}(\in \infty) \\
\uparrow &
\end{align*}
\]

\[
\begin{align*}
&\kappa^{b'}(\neq \infty) \rightarrow \kappa^{b'}(\leq^*) \\
\uparrow &
\end{align*}
\]

\[
\begin{align*}
&\kappa^{b',h}(\in^*) \rightarrow \kappa^{b',h}(\in \infty) = \kappa^b(\neq \infty) \rightarrow \kappa^b(\leq^*)
\end{align*}
\]
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In Brendle et al. [2018] it is shown that \( \kappa^+ < b_{\kappa}^{\kappa,h}(\infty) \) and \( d_{\kappa}^{\kappa,h}(\infty) < 2^\kappa \) is consistent for any increasing cofinal \( h \in \kappa \kappa \), using an iteration of generalised Localisation forcing. Furthermore it is shown that \( d_{\kappa}^{\kappa,pow}(\infty) < d_{\kappa}^{\kappa,id}(\infty) \) is consistent, where \( id : \alpha \mapsto |\alpha| \) and \( pow : \alpha \mapsto 2^{|\alpha|} \), using a product of the generalised Sacks forcing from Kanamori [1980].

In [vdV] we showed that there exists a set \( \{ h_\xi \in \kappa \kappa \mid \xi < \kappa \} \) such that for any sequence of cardinals \( \langle \kappa_\xi \mid \xi < \kappa \rangle \) with \( \kappa_\xi \geq \kappa^+ \) for each \( \xi \) it is consistent that \( \forall \xi \in \kappa \left( d_{\kappa}^{\kappa,h_\xi}(\infty) = \kappa_\xi \right) \). The forcing used is a product of Sacks-like forcings.

The same consistency generalises to increasing cofinal \( b \in \kappa \kappa \): there exists a set \( \{ h_\xi \in \prod b \mid \xi < \kappa \} \) such that \( \forall \xi \in \kappa \left( d_{\kappa}^{b,h_\xi}(\infty) = \kappa_\xi \right) \).
Let \( h \in \kappa \kappa \) be an increasing cofinal cardinal function. The conditions of the forcing \( S_h^\kappa \) are trees \( T \subseteq \kappa \kappa \) that satisfy the following properties:

(i) for any \( u \in T \) there exists splitting \( v \in T \) such that \( u \subseteq v \),

(ii) if \( \gamma < \kappa \) and \( \langle u_\alpha \mid \alpha < \gamma \rangle \in \gamma T \) are splitting nodes with \( u_\alpha \subseteq u_\beta \) for \( \alpha < \beta \), then \( u = \bigcup_{\alpha < \gamma} u_\alpha \in T \) and \( u \) is splitting,

(iii) if \( u \in \text{Split}_\alpha(T) \), then \( u \) is an \( h(\alpha) \)-splitting node in \( T \).

We say that \( T \leq S \) iff \( T \subseteq S \) and for every splitting \( u \in T \), either \( \text{suc}(u, T) = \text{suc}(u, S) \) or \( |\text{suc}(u, T)| < |\text{suc}(u, S)| \).

The \( \leq \kappa \)-support product of forcings \( S_h^\kappa \) is \( \kappa \)-closed, satisfies generalised fusion, and has the generalised \( h \)-Sacks property.
To separate $b^{b,h}_\omega(\in^*)$, typically creature forcings with a $\lim\inf$-norm are used. These resemble tree forcings that split everywhere above the stem, e.g. Laver forcing.

However, due to limit ordinals being present in $\kappa$, properties such as pure decision are not available. This makes separating cardinals of the form $b^{b,h}_\kappa(\in^*)$ significantly harder.
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○ Consistency results for $b_{\kappa}^{b,h}(\in^\infty)$ and $\mathcal{d}_{\kappa}^{b,h}(\in^\infty)$
Let $\mathbb{P}_{\kappa}^{b,h}$ be a forcing with trees $T$ on $\text{Loc}_{h}^{b}$ as conditions, i.e. $u \in T$ implies $u : \alpha \to [\kappa]^{<\kappa}$ s.t. $u(\xi) \in [b(\xi)]^{<h(\xi)}$ for each $\xi < \alpha$.

If $u \in T$ with $\alpha = \text{ot}(u)$, let $\|u\|_T$ be the least $\nu < \kappa$ such that there exists $A \in [b(\alpha)]^\nu$ such that $A \not\subseteq A'$ for all $A' \in \text{suc}(u,T)$.

Let $T \in \mathbb{P}_{\kappa}^{b,h}$ iff

(i) for all $u \in T$, $\nu < \kappa$ there is $v \in T$ with $u \subseteq v$ and $\nu \leq \|v\|_T$,
(ii) If $\langle u_\xi \mid \xi < \gamma \rangle$ is a sequence of splitting nodes and $u_\xi \subseteq u'_\xi$ for $\xi < \xi'$, then $\bigcup_{\xi < \gamma} u_\xi$ splits in $T$,
(iii) if $u \in \text{Split}_\alpha(T)$, then $\max \{ |\alpha|, 2 \} \leq \|u\|_T$.

Let $S \leq_{\mathbb{P}_{\kappa}^{b,h}} T$ if $S \subseteq T$ and for each $s \in S$ either $\text{suc}(s,S) = \text{suc}(s,T)$ or $\|s\|_S < \|s\|_T$. 

Consistency results for $b^{b,h}_{\kappa}(\infty)$ and $\delta^{b,h}_{\kappa}(\infty)$

**Theorem**
If $b \in \kappa^\kappa$, then $\text{cov}(\mathcal{M}_\kappa) = b^{\kappa,h}_{\kappa}(\infty) \leq \delta^{\kappa}_{\kappa}(\leq^*) < b^{b,h}_{\kappa}(\infty)$ is consistent.

$\mathbb{P}^{b,h}_{\kappa}$ is $<\kappa$-closed, has fusion and is $\kappa^\kappa$-bounding. Moreover, the $\leq\kappa$-support iteration of $\mathbb{P}^{b,h}_{\kappa}$ is $\kappa^\kappa$-bounding as well. Hence, forcing with $\mathbb{P}^{b,h}_{\kappa}$ increases the size of $b^{b,h}_{\kappa}(\infty)$ but keeps $\text{cov}(\mathcal{M}_\kappa)$ and $\delta^{\kappa}_{\kappa}(\leq^*)$ small.

The goal is to use this forcing in a similar way to the methods described in Cardona et al. [2021] to separate cardinals of the form $b^{b,h}_{\kappa}(\infty)$ for different $b \in \kappa^\kappa$.

Separating cardinals of the form $\delta^{b,h}_{\kappa}(\infty)$ has similar problems as separating cardinals of the form $b^{b,h}_{\kappa}(\in^*)$.


