An Overview of (Anti-)Localisation Cardinals on Products of Discrete Spaces

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The **Baire space** is the set ${}^{\omega}\omega = \{f \mid f : \omega \to \omega\}$, with clopen sets $[s] = \{f \in {}^{\omega}\omega \mid s \subseteq f\}$ for $s \in {}^{<\omega}\omega$ generating the topology. Let $\mathcal{N}$ and $\mathcal{M}$ resp. be the σ-ideals of Lebesgue null and meagre sets.

$$\begin{aligned}
\mathrm{cov}(\mathcal{I}) &= \min\left\{|\mathcal{C}| \;\middle|\; \mathcal{C} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{C} = {}^{\omega}\omega\right\},\\
\mathrm{non}(\mathcal{I}) &= \min\left\{|N| \;\middle|\; N \subseteq {}^{\omega}\omega \text{ and } N \notin \mathcal{I}\right\},\\
\mathrm{add}(\mathcal{I}) &= \min\left\{|\mathcal{A}| \;\middle|\; \mathcal{A} \subseteq \mathcal{I} \text{ and } \bigcup \mathcal{A} \notin \mathcal{I}\right\},\\
\mathrm{cof}(\mathcal{I}) &= \min\left\{|\mathcal{J}| \;\middle|\; \mathcal{J} \subseteq \mathcal{I} \text{ and } \forall X \in \mathcal{I} \exists Y \in \mathcal{J} (X \subseteq Y)\right\}.
\end{aligned}$$

$$\begin{tikzcd}
 & \mathrm{cov}(\mathcal{N}) \arrow[r] & \mathbf{non}(\mathcal{M}) \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \mathbf{cof}(\mathcal{N}) \arrow[r] & 2^{\aleph_0} \\
 & & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & & \\
\aleph_1 \arrow[r] & \mathbf{add}(\mathcal{N}) \arrow[r] \arrow[uu] & \mathrm{add}(\mathcal{M}) \arrow[r] \arrow[u] & \mathbf{cov}(\mathcal{M}) \arrow[r] \arrow[u] & \mathrm{non}(\mathcal{N}) \arrow[uu] &
\end{tikzcd}$$

Let κ be uncountable, then ${}^\kappa\kappa$ is a **generalised Baire space**. We say $f \in {}^\kappa\kappa$ are κ-reals. If κ is strongly inaccessible, we can generalise the middle part of the Cichoń Diagram:

$$\begin{array}{ccccccc} & & \mathrm{non}(\mathcal{M}_\kappa) & \longrightarrow & \mathrm{cof}(\mathcal{M}_\kappa) & \longrightarrow & 2^\kappa \\ & & \uparrow & & \uparrow & & \\ & & \mathfrak{b}_\kappa & \longrightarrow & \mathfrak{d}_\kappa & & \\ & & \uparrow & & \uparrow & & \\ \kappa^+ & \longrightarrow & \mathrm{add}(\mathcal{M}_\kappa) & \longrightarrow & \mathrm{cov}(\mathcal{M}_\kappa) & & \end{array}$$

There is no Lebesgue measure on ${}^\kappa\kappa$, so there is no generalisation of $\mathcal{N}$ to ${}^\kappa\kappa$. We can generalise $\mathrm{add}(\mathcal{N})$ and $\mathrm{cof}(\mathcal{N})$ using a combinatorial definition instead.

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Let κ be regular strong limit and let h, b be increasing cardinal function with domain κ. We define the bounded space $\prod b = \prod_{\alpha\in\kappa} b(\alpha) = \{f : \kappa \to \mathrm{Ord} \mid \forall\alpha < \kappa(f(\alpha) < b(\alpha))\}$.

Let φ with $\mathrm{dom}(\varphi) = \kappa$ be an (h, b)-**slalom** if $\varphi(\alpha) \in [b(\alpha)]^{<h(\alpha)}$ for all $\alpha \in \kappa$, and Loc^b_h be the set of (h, b)-slaloms.

For $f, g \in \prod b$ and $\varphi \in \mathrm{Loc}^b_h$, we say

- $f \in^* \varphi$ iff $f(\alpha) \in \varphi(\alpha)$ for almost all $\alpha < \kappa$,
- $f \in^\infty \varphi$ iff $f(\alpha) \in \varphi(\alpha)$ for cofinally many $\alpha < \kappa$,
- $f =^\infty g$ iff $f(\alpha) = g(\alpha)$ for cofinally many $\alpha < \kappa$,
- $f \leq^* g$ iff $f(\alpha) \leq g(\alpha)$ for almost all $\alpha < \kappa$.

Let $\mathscr{R} = \langle X, Y, R\rangle$ be a **relational system**, i.e. $R \subseteq X \times Y$. Let $\|\mathscr{R}\| = \min\{|\mathcal{Z}| \mid \mathcal{Z} \subseteq Y \text{ and } \forall a \in X \exists b \in \mathcal{Z}(a\ R\ b)\}$ be the **norm** of $\mathscr{R}$. The **dual** $\not{R}^{-1} = \{(b, a) \in Y \times X \mid (a, b) \notin R\}$ provides a **dual relational system** $\mathscr{R}^{\perp} = \left\langle Y, X, \not{R}^{-1}\right\rangle$.

Given $\mathscr{R} = \langle X, Y, R\rangle$ and $\mathscr{R}' = \langle X', Y', R'\rangle$, a **Tukey connection** from $\mathscr{R}$ to $\mathscr{R}'$ is a pair $\rho_- : X \to X'$ and $\rho_+ : Y' \to Y$ such that for any $x \in X$ and $y' \in Y'$ with $(\rho_-(x), y') \in R'$ we also have $(x, \rho_+(y')) \in R$. We let $\mathscr{R} \preceq \mathscr{R}'$ denote that there exists a Tukey connection from $\mathscr{R}$ to $\mathscr{R}'$, and $\mathscr{R} \equiv \mathscr{R}'$ that $\mathscr{R} \preceq \mathscr{R}' \preceq \mathscr{R}$.

Lemma

If $\mathscr{R} \preceq \mathscr{R}'$, then $\|\mathscr{R}\| \leq \|\mathscr{R}'\|$ and $\left\|\mathscr{R}'^{\perp}\right\| \leq \left\|\mathscr{R}^{\perp}\right\|$.

We define the following relational systems:

$$\mathscr{L}_h^b = \left\langle \prod b, \mathrm{Loc}_h^b, \in^* \right\rangle \quad \text{(localisation)}$$
$$\mathscr{AL}_h^b = \left\langle \prod b, \mathrm{Loc}_h^b, \in^\infty \right\rangle \quad \text{(anti-localisation)}$$
$$\mathscr{ED}^b = \langle \prod b, \prod b, \neq^\infty \rangle \quad \text{(eventually different / cofinally equal)}$$
$$\mathscr{D}^b = \langle \prod b, \prod b, \leq^* \rangle \quad \text{(dominating / unbounding)}$$

with the norms:

$$\left\| \mathscr{L}_h^b \right\| = \mathfrak{d}_\kappa^{b,h}(\in^*) \qquad \left\| \mathscr{L}_h^{b\perp} \right\| = \mathfrak{b}_\kappa^{b,h}(\in^*)$$
$$\left\| \mathscr{AL}_h^b \right\| = \mathfrak{d}_\kappa^{b,h}(\in^\infty) \qquad \left\| \mathscr{AL}_h^{b\perp} \right\| = \mathfrak{b}_\kappa^{b,h}(\in^\infty)$$
$$\left\| \mathscr{ED}^b \right\| = \mathfrak{d}_\kappa^{b}(\neq^\infty) \qquad \left\| \mathscr{ED}^{b\perp} \right\| = \mathfrak{b}_\kappa^{b}(\neq^\infty)$$
$$\left\| \mathscr{D}^b \right\| = \mathfrak{d}_\kappa^{b}(\leq^*) \qquad \left\| \mathscr{D}^{b\perp} \right\| = \mathfrak{b}_\kappa^{b}(\leq^*)$$

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Let $\overline{\omega} : n \mapsto \omega$ for all $n \in \omega$ and let $h \in {}^{\omega}\omega$ be cofinal.

Theorem *Bartoszyński [1987]*
$\mathfrak{b}^{\overline{\omega},h}_{\omega}(\in^*) = \mathrm{add}(\mathcal{N})$ and
$\mathfrak{d}^{\overline{\omega},h}_{\omega}(\in^*) = \mathrm{cof}(\mathcal{N})$.

Theorem *Bartoszyński [1987] or Bartoszyński and Judah [1995]*
$\mathfrak{d}^{\overline{\omega},h}_{\omega}(\in^\infty) = \mathfrak{b}^{\overline{\omega}}_{\omega}(\neq^\infty) = \mathrm{non}(\mathcal{M})$ and
$\mathfrak{b}^{\overline{\omega},h}_{\omega}(\in^\infty) = \mathfrak{d}^{\overline{\omega}}_{\omega}(\neq^\infty) = \mathrm{cov}(\mathcal{M})$.

Proposition
$\mathfrak{b}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^*) = \mathfrak{d}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^\infty) = \mathfrak{b}^{\overline{\omega}}_{\omega}(\leq^*) = \mathfrak{b}$ and
$\mathfrak{d}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^*) = \mathfrak{b}^{\overline{\omega},\overline{\omega}}_{\omega}(\in^\infty) = \mathfrak{d}^{\overline{\omega}}_{\omega}(\leq^*) = \mathfrak{d}$.

$$
\begin{tikzcd}
 & & & 2^{\aleph_0} \\
\mathrm{cov}(\mathcal{N}) \arrow[r] & \boxed{\begin{array}{c}\mathbf{non}(\mathcal{M}) \\ \mathfrak{d}_{\omega}^{\overline{\omega},h}(\in^{\infty}) \\ \mathfrak{b}_{\omega}^{\overline{\omega}}(\neq^{\infty})\end{array}} \arrow[r] & \mathrm{cof}(\mathcal{M}) \arrow[r] & \boxed{\begin{array}{c}\mathbf{cof}(\mathcal{N}) \\ \mathfrak{d}_{\omega}^{\overline{\omega},h}(\in^{*})\end{array}} \arrow[u] \\
 & \boxed{\begin{array}{c}\mathfrak{b} \\ \mathfrak{b}_{\omega}^{\overline{\omega},\overline{\omega}}(\in^{*}) \\ \mathfrak{d}_{\omega}^{\overline{\omega},\overline{\omega}}(\in^{\infty})\end{array}} \arrow[u] \arrow[r] & \boxed{\begin{array}{c}\mathfrak{b}_{\omega}^{\overline{\omega},\overline{\omega}}(\in^{\infty}) \\ \mathfrak{d}_{\omega}^{\overline{\omega},\overline{\omega}}(\in^{*}) \\ \mathfrak{d}\end{array}} \arrow[u] & \\
\boxed{\begin{array}{c}\mathfrak{b}_{\omega}^{\overline{\omega},h}(\in^{*}) \\ \mathbf{add}(\mathcal{N})\end{array}} \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r] & \boxed{\begin{array}{c}\mathfrak{d}_{\omega}^{\overline{\omega}}(\neq^{\infty}) \\ \mathfrak{b}_{\omega}^{\overline{\omega},h}(\in^{\infty}) \\ \mathbf{cov}(\mathcal{M})\end{array}} \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) \arrow[uu] \\
\aleph_1 \arrow[u] & & &
\end{tikzcd}
$$

Theorem *Cardona et al. [2021]*

It is consistent that there exist $h_\xi, b_\xi \in {}^\omega\omega$ for each $\xi < \mathfrak{c}$ and a strictly increasing sequence of cardinals $\langle \kappa_\xi \mid \xi < \mathfrak{c} \rangle$ such that $\mathfrak{b}^{b,h}_\omega(\in^*) = \mathfrak{d}^{b,h}_\omega(\in^*) = \mathfrak{b}^{b,h}_\omega(\in^\infty) = \mathfrak{d}^{b,h}_\omega(\in^\infty) = \kappa_\xi$.

The proof is the culmination of the investigation into these cardinals using creature forcings, originating from Goldstern and Shelah [1993] and improved by Kellner and Shelah, and later in connection with Yorioka ideals in several papers by Kamo, Osuga, Brendle, Mejía, Klausner and Cardona.

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For some choices of b and h, the bounded (anti-)localisation cardinals may be trivial.

Lemma
$\mathfrak{d}^{b,h}_{\kappa}(\in^*) = 1$ iff $b <^* h$, which implies $\mathfrak{b}^{b,h}_{\kappa}(\in^*)$ is undefined.
$\mathfrak{d}^{b,h}_{\kappa}(\in^\infty) = 1$ iff $b <^\infty h$, which implies $\mathfrak{b}^{b,h}_{\kappa}(\in^\infty)$ is undefined.

Lemma *Cardona and Mejía [2019] & Goldstern and Shelah [1993] ($\kappa = \omega$)*
If $\lambda < \kappa$ exists and is minimal s.t. $D_\lambda = \{\alpha \in \kappa \mid h(\alpha) = \lambda\}$ is cofinal in κ, then $\mathfrak{b}^{b,h}_{\kappa}(\in^*) = \lambda$ and $2^\kappa \leq \mathfrak{d}^{b,h}_{\kappa}(\in^*)$. If no such λ exists, $\kappa^+ \leq \mathfrak{b}^{b,h}_{\kappa}(\in^*)$, and if also $b \leq 2^\kappa$, then $\mathfrak{d}^{b,h}_{\kappa}(\in^*) \leq 2^\kappa$.

Trivial cases

Let increasing $f : \kappa \to \mathrm{Ord}$ be **continuous** at $\gamma \in \kappa$ if $f(\gamma) = \bigcup_{\alpha<\gamma} f(\alpha)$. We call f stationarily continuous there exists S stationary in κ s.t. f is continuous at all limit $\gamma \in S$.

Lemma

For $\lambda < \kappa$ let
$D_\lambda = \{\alpha \in \kappa \mid b(\alpha) \leq \lambda\} \cup \{\alpha \in \kappa \mid h(\alpha) = b(\alpha) \wedge \mathrm{cf}(b(\alpha)) \leq \lambda\}$.

(i) If $\lambda < \kappa$ exists and is minimal s.t. D_λ is cofinal in κ, then $\mathfrak{d}_\kappa^{b,h}(\in^\infty) = \lambda$.

(ii) If all D_λ are bounded, b is stat.cont., then $\mathfrak{d}_\kappa^{b,h}(\in^\infty) = \kappa$.

(iii) If all D_λ are bounded, b is not stat.cont., then $\kappa^+ \leq \mathfrak{d}_\kappa^{b,h}(\in^\infty)$.

A dual result for the relation between $\mathfrak{b}_\kappa^{b,h}(\in^\infty)$ and 2^κ is not known yet.

If $h \leq^* h'$ and $b \geq^* b'$, then:

$$\mathscr{AL}^{b'}_{h'} \preceq \mathscr{AL}^{b}_{h} \preceq \mathscr{L}^{b}_{h} \qquad \mathscr{AL}^{b'}_{h'} \preceq \mathscr{L}^{b'}_{h'} \preceq \mathscr{L}^{b}_{h}$$

$$\mathscr{ED}^{b} \preceq \mathscr{ED}^{b'} \qquad \mathscr{ED}^{b} \preceq \mathscr{D}^{b}$$

$$\mathfrak{d}^{b',h'}_{\kappa}(\in^\infty) \leq \mathfrak{d}^{b',h'}_{\kappa}(\in^*) \leq \mathfrak{d}^{b,h}_{\kappa}(\in^*) \quad \mathfrak{b}^{b,h}_{\kappa}(\in^*) \leq \mathfrak{b}^{b',h'}_{\kappa}(\in^*) \leq \mathfrak{b}^{b',h'}_{\kappa}(\in^\infty)$$

$$\mathfrak{d}^{b',h'}_{\kappa}(\in^\infty) \leq \mathfrak{d}^{b,h}_{\kappa}(\in^\infty) \leq \mathfrak{d}^{b,h}_{\kappa}(\in^*) \quad \mathfrak{b}^{b,h}_{\kappa}(\in^*) \leq \mathfrak{b}^{b,h}_{\kappa}(\in^\infty) \leq \mathfrak{b}^{b',h'}_{\kappa}(\in^\infty)$$

$$\mathfrak{d}^{b}_{\kappa}(\neq^\infty) \leq \mathfrak{d}^{b'}_{\kappa}(\neq^\infty) \leq \mathfrak{d}^{b'}_{\kappa}(\leq^*) \qquad \mathfrak{b}^{b}_{\kappa}(\leq^*) \leq \mathfrak{b}^{b}_{\kappa}(\neq^\infty) \leq \mathfrak{b}^{b'}_{\kappa}(\neq^\infty)$$

Let $\overline{\kappa} : \alpha \mapsto \kappa$ for all $\alpha \in \kappa$.

The relation between eventual difference and the meagre ideal generalise to strongly inaccessible κ.

Theorem *Landver [1992] and Blass et al. [2005]*
$\mathfrak{d}^{\overline{\kappa}}_{\kappa}(\neq^\infty) = \mathrm{cov}(\mathcal{M}_\kappa)$ and $\mathfrak{b}^{\overline{\kappa}}_{\kappa}(\neq^\infty) = \mathrm{non}(\mathcal{M}_\kappa)$.

Theorem *Brendle et al. [2018]*
$\max\left\{\mathrm{non}(\mathcal{M}_\kappa), \mathfrak{d}^{\overline{\kappa}}_{\kappa}(\leq^*)\right\} = \mathrm{cof}(\mathcal{M}_\kappa) \leq \mathfrak{d}^{\overline{\kappa},h}_{\kappa}(\in^*)$ and
$\min\left\{\mathrm{cov}(\mathcal{M}_\kappa), \mathfrak{b}^{\overline{\kappa}}_{\kappa}(\leq^*)\right\} = \mathrm{add}(\mathcal{M}_\kappa) \geq \mathfrak{b}^{\overline{\kappa},h}_{\kappa}(\in^*)$.

Proposition
If $h \in {}^\kappa\kappa$, then $\mathfrak{d}^{\overline{\kappa},h}_{\kappa}(\in^\infty) = \mathfrak{b}^{\overline{\kappa}}_{\kappa}(\neq^\infty)$ and $\mathfrak{b}^{\overline{\kappa},h}_{\kappa}(\in^\infty) = \mathfrak{d}^{\overline{\kappa}}_{\kappa}(\neq^\infty)$.

We will state a more general result.

Say that b **overshadows** h if there exists an interval partition $\langle I_\alpha \mid \alpha < \kappa\rangle$ of κ with $|I_\alpha| = h(\alpha)$ for each $\alpha \in \kappa$ such that $b(\alpha) = b(\xi) = b(\alpha)^{h(\alpha)}$ for all $\xi \in I_\alpha$ and $\alpha \in \kappa$.

Theorem

If b overshadows h, then $\mathfrak{d}^{b,h}_\kappa(\in^\infty) = \mathfrak{b}^b_\kappa(\neq^\infty)$ and $\mathfrak{b}^{b,h}_\kappa(\in^\infty) = \mathfrak{d}^b_\kappa(\neq^\infty)$.

Note that $\overline{\kappa}$ overshadows any $h \in {}^\kappa\kappa$. In particular, the cardinalities of $\mathfrak{d}^{\overline{\kappa},h}_\kappa(\in^\infty)$ and $\mathfrak{b}^{\overline{\kappa},h}_\kappa(\in^\infty)$ do not depend on the choice of $h \in {}^\kappa\kappa$.

Nontrivial cases

Let $h \leq h' \leq \overline{\kappa}$ be increasing cofinal and $h \in {}^\kappa\kappa$. If $h' =^* b$, the dotted lines are equality.

$$
\begin{tikzcd}
\mathfrak{d}_\kappa^{\overline{\kappa}}(\leq^*) \arrow[rr] \arrow[rr, dotted, no head, shift right] \arrow[dr, dotted, no head] & & \mathfrak{d}_\kappa^{\overline{\kappa},h'}(\in^*) \arrow[r] & \mathfrak{d}_\kappa^{\overline{\kappa},h}(\in^*) \\
& \mathfrak{b}_\kappa^{\overline{\kappa},h'}(\in^\infty) \arrow[ur, dotted, no head] & & \\
\boxed{\begin{matrix} \mathrm{cov}(\mathcal{M}_\kappa) \\ \mathfrak{b}_\kappa^{\overline{\kappa},h}(\in^\infty) \\ \mathfrak{d}_\kappa^{\overline{\kappa}}(\neq^\infty) \end{matrix}} \arrow[uu] \arrow[ur] & & & \boxed{\begin{matrix} \mathfrak{b}_\kappa^{\overline{\kappa}}(\neq^\infty) \\ \mathfrak{d}_\kappa^{\overline{\kappa},h}(\in^\infty) \\ \mathrm{non}(\mathcal{M}_\kappa) \end{matrix}} \arrow[uu] \\
& & \mathfrak{d}_\kappa^{\overline{\kappa},h'}(\in^\infty) \arrow[uuu] \arrow[ur] \arrow[dr, dotted, no head] & \\
\mathfrak{b}_\kappa^{\overline{\kappa},h}(\in^*) \arrow[uu] \arrow[r] & \mathfrak{b}_\kappa^{\overline{\kappa},h'}(\in^*) \arrow[uuu] \arrow[rr] \arrow[rr, dotted, no head, shift left] \arrow[ur, dotted, no head] & & \mathfrak{b}_\kappa^{\overline{\kappa}}(\leq^*) \arrow[uu] \arrow[uuuulll, bend left]
\end{tikzcd}
$$

Let $h \leq b' \leq b \in {}^\kappa\kappa$ be increasing cofinal and b overshadows h.

$$
\begin{tikzcd}
\mathfrak{b}_\kappa^{b}(\leq^*) \arrow[r] & \mathfrak{b}_\kappa^{b}(\neq^\infty) = \mathfrak{d}_\kappa^{b,h}(\in^\infty) \arrow[r] & \mathfrak{d}_\kappa^{b,h}(\in^*) \\
\mathfrak{b}_\kappa^{b'}(\leq^*) \arrow[r] & \mathfrak{b}_\kappa^{b'}(\neq^\infty) \arrow[u] & \\
 & \mathfrak{d}_\kappa^{b',h}(\in^\infty) \arrow[r] \arrow[u] & \mathfrak{d}_\kappa^{b',h}(\in^*) \arrow[uu]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathfrak{b}_\kappa^{b',h}(\in^*) \arrow[r] & \mathfrak{b}_\kappa^{b',h}(\in^\infty) & \\
 & \mathfrak{d}_\kappa^{b'}(\neq^\infty) \arrow[r] \arrow[u] & \mathfrak{d}_\kappa^{b'}(\leq^*) \\
\mathfrak{b}_\kappa^{b,h}(\in^*) \arrow[r] \arrow[uu] & \mathfrak{b}_\kappa^{b,h}(\in^\infty) = \mathfrak{d}_\kappa^{b}(\neq^\infty) \arrow[r] \arrow[u] & \mathfrak{d}_\kappa^{b}(\leq^*)
\end{tikzcd}
$$

Contents

Consistency results for $\mathfrak{b}_\kappa^{b,h}(\in^*)$ and $\mathfrak{d}_\kappa^{b,h}(\in^*)$

In Brendle et al. [2018] it is shown that $\kappa^+ < \mathfrak{b}_\kappa^{\overline{\kappa},h}(\in^*)$ and $\mathfrak{d}_\kappa^{\overline{\kappa},h}(\in^*) < 2^\kappa$ is consistent for any increasing cofinal $h \in {}^\kappa\kappa$, using an iteration of generalised Localisation forcing. Furthermore it is shown that $\mathfrak{d}_\kappa^{\overline{\kappa},\mathsf{pow}}(\in^*) < \mathfrak{d}_\kappa^{\overline{\kappa},\mathsf{id}}(\in^*)$ is consistent, where $\mathsf{id} : \alpha \mapsto |\alpha|$ and $\mathsf{pow} : \alpha \mapsto 2^{|\alpha|}$, using a product of the generalised Sacks forcing from Kanamori [1980].

In [vdV] we showed that there exists a set $\{h_\xi \in {}^\kappa\kappa \mid \xi < \kappa\}$ such that for any sequence of cardinals $\langle \kappa_\xi \mid \xi < \kappa \rangle$ with $\kappa_\xi \geq \kappa^+$ for each ξ it is consistent that $\forall \xi \in \kappa \left(\mathfrak{d}_\kappa^{\overline{\kappa},h_\xi}(\in^*) = \kappa_\xi \right)$. The forcing used is a product of Sacks-like forcings.

The same consistency generalises to increasing cofinal $b \in {}^\kappa\kappa$: there exists a set $\{h_\xi \in \prod b \mid \xi < \kappa\}$ such that $\forall \xi \in \kappa \left(\mathfrak{d}_\kappa^{b,h_\xi}(\in^*) = \kappa_\xi \right)$.

Let $h \in {}^{\kappa}\kappa$ be an increasing cofinal cardinal function. The conditions of the forcing $\mathbb{S}^h_\kappa$ are trees $T \subseteq {}^{<\kappa}\kappa$ that satisfy the following properties:

(i) for any $u \in T$ there exists splitting $v \in T$ such that $u \subseteq v$,

(ii) if $\gamma < \kappa$ and $\langle u_\alpha \mid \alpha < \gamma \rangle \in {}^{\gamma}T$ are splitting nodes with $u_\alpha \subseteq u_\beta$ for $\alpha < \beta$, then $u = \bigcup_{\alpha<\gamma} u_\alpha \in T$ and u is splitting,

(iii) if $u \in \mathrm{Split}_\alpha(T)$, then u is an $h(\alpha)$-splitting node in T.

We say that $T \leq S$ iff $T \subseteq S$ and for every splitting $u \in T$, either $\mathrm{suc}(u,T) = \mathrm{suc}(u,S)$ or $|\mathrm{suc}(u,T)| < |\mathrm{suc}(u,S)|$.

The $\leq\kappa$-support product of forcings $\mathbb{S}^h_\kappa$ is $<\kappa$-closed, satisfies generalised fusion, and has the generalised h-Sacks property.

To separate $\mathfrak{b}_{\omega}^{b,h}(\in^*)$, typically creature forcings with a $\liminf$-norm are used. These resemble tree forcings that split everywhere above the stem, e.g. Laver forcing.

However, due to limit ordinals being present in κ, properties such as pure decision are not available. This makes separating cardinals of the form $\mathfrak{b}_{\kappa}^{b,h}(\in^*)$ significantly harder.

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Let $\mathbb{P}_\kappa^{b,h}$ be a forcing with trees T on Loc_h^b as conditions, i.e. $u \in T$ implies $u : \alpha \to [\kappa]^{<\kappa}$ s.t. $u(\xi) \in [b(\xi)]^{<h(\xi)}$ for each $\xi < \alpha$.

If $u \in T$ with $\alpha = \mathrm{ot}(u)$, let $\|u\|_T$ be the least $\nu < \kappa$ such that there exists $A \in [b(\alpha)]^\nu$ such that $A \not\subseteq A'$ for all $A' \in \mathrm{suc}(u, T)$.

Let $T \in \mathbb{P}_\kappa^{b,h}$ iff

(i) for all $u \in T$, $\nu < \kappa$ there is $v \in T$ with $u \subseteq v$ and $\nu \leq \|v\|_T$,

(ii) If $\langle u_\xi \mid \xi < \gamma \rangle$ is a sequence of splitting nodes and $u_\xi \subseteq u'_\xi$ for $\xi < \xi'$, then $\bigcup_{\xi<\gamma} u_\xi$ splits in T,

(iii) if $u \in \mathrm{Split}_\alpha(T)$, then $\max\{|\alpha|, 2\} \leq \|u\|_T$.

Let $S \leq_{\mathbb{P}_\kappa^{b,h}} T$ if $S \subseteq T$ and for each $s \in S$ either $\mathrm{suc}(s, S) = \mathrm{suc}(s, T)$ or $\|s\|_S < \|s\|_T$.

Theorem

If $b \in {}^\kappa\kappa$, then $\operatorname{cov}(\mathcal{M}_\kappa) = \mathfrak{b}^{\overline{\kappa},h}_\kappa(\in^\infty) \leq \mathfrak{d}^{\overline{\kappa}}_\kappa(\leq^*) < \mathfrak{b}^{b,h}_\kappa(\in^\infty)$ is consistent.

$\mathbb{P}^{b,h}_\kappa$ is $<\kappa$-closed, has fusion and is ${}^\kappa\kappa$-bounding. Moreover, the $\leq\kappa$-support iteration of $\mathbb{P}^{b,h}_\kappa$ is ${}^\kappa\kappa$-bounding as well. Hence, forcing with $\mathbb{P}^{b,h}_\kappa$ increases the size of $\mathfrak{b}^{b,h}_\kappa(\in^\infty)$ but keeps $\operatorname{cov}(\mathcal{M}_\kappa)$ and $\mathfrak{d}^{\overline{\kappa}}_\kappa(\leq^*)$ small.

The goal is to use this forcing in a similar way to the methods described in Cardona et al. [2021] to separate cardinals of the form $\mathfrak{b}^{b,h}_\kappa(\in^\infty)$ for different $b \in {}^\kappa\kappa$.

Separating cardinals of the form $\mathfrak{d}^{b,h}_\kappa(\in^\infty)$ has similar problems as separating cardinals of the form $\mathfrak{b}^{b,h}_\kappa(\in^*)$.

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