A Hierarchy of Compactness Cardinals below Vopěnka's Principle

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Motivation

Vopěnka's Principle (VP) is the statement

> *"For every language τ, if $\mathcal{K}$ is a proper class of τ-structures, then there are distinct $\mathcal{A}, \mathcal{B} \in \mathcal{K}$ s.t. there is an elementary embedding $j : \mathcal{A} \preccurlyeq \mathcal{B}$."*

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Con(ZFC + there is an almost huge cardinal) $\rightarrow$ Con(ZFC + VP).

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The proof needs the following:

Lemma

There is a sentence, known as *Magidor's* Φ, of second-order logic in the language $\{E\}$ s.t. $(M, E^M) \models \Phi$ iff there is a limit ordinal α s.t. $(M, E^M) \cong (V_\alpha, \in)$.

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Theorem (Makowsky [4])

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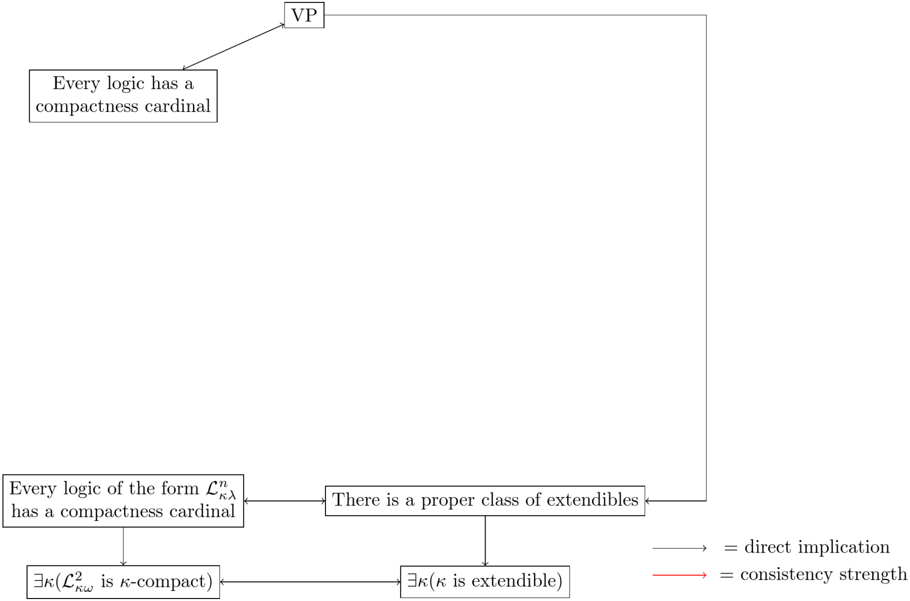
Summarizing what we stated until now, we get the following picture:



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The following are equivalent:

(i) Vopěnka's Principle.

(ii) For every n: there is a $C^{(n)}$-extendible cardinal.

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$C^{(n)}$ is the following class:

$$C^{(n)} := \{\alpha \in Ord\colon (V_\alpha, \in) \preccurlyeq_n (V, \in)\}.$$

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κ is *$C^{(n)}$-extendible* iff it is α-$C^{(n)}$-extendible for every $\alpha > \kappa$.

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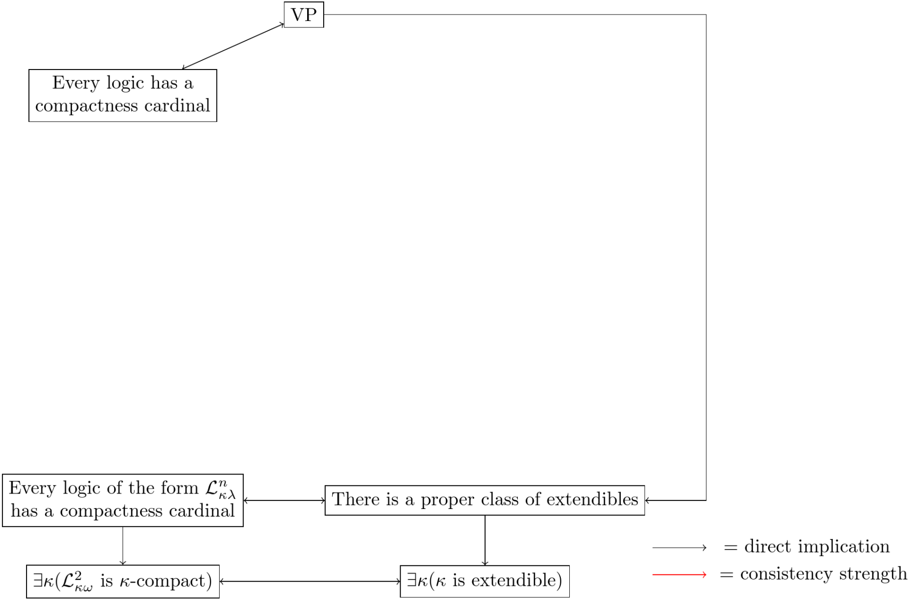
Now consider the smallest $C^{(2)}$-extendible cardinal κ, should it exist. By items (c)+(d) above, $\{x < \kappa \colon x \text{ is extendible}\}$ is unbounded below κ. Then by item (a)+(b),

$$V_\kappa \models \text{ZFC} \wedge \text{ there is a proper class of extendibles} \\ \wedge \neg\exists x(x \text{ is } C^{(2)}\text{-extendible}).$$

Now we can update our picture: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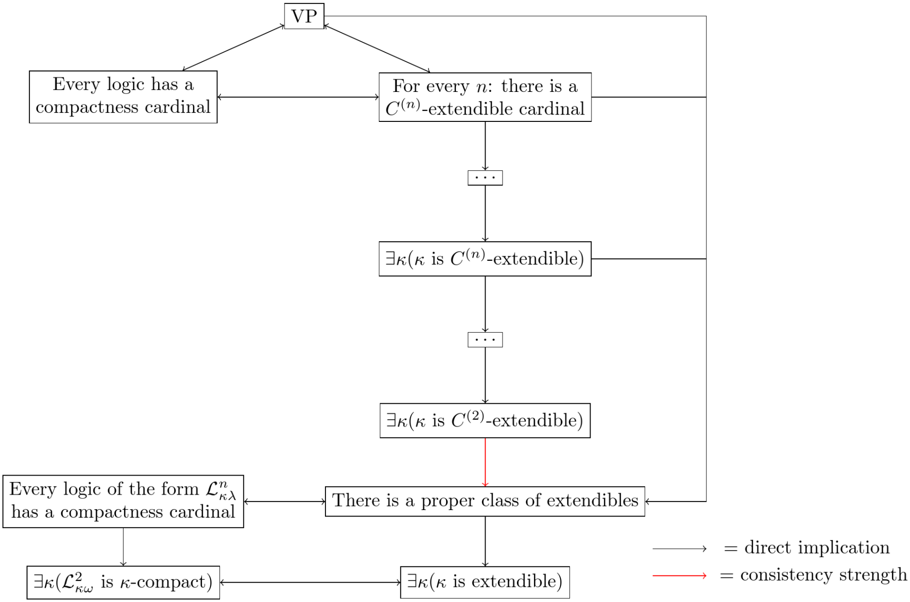
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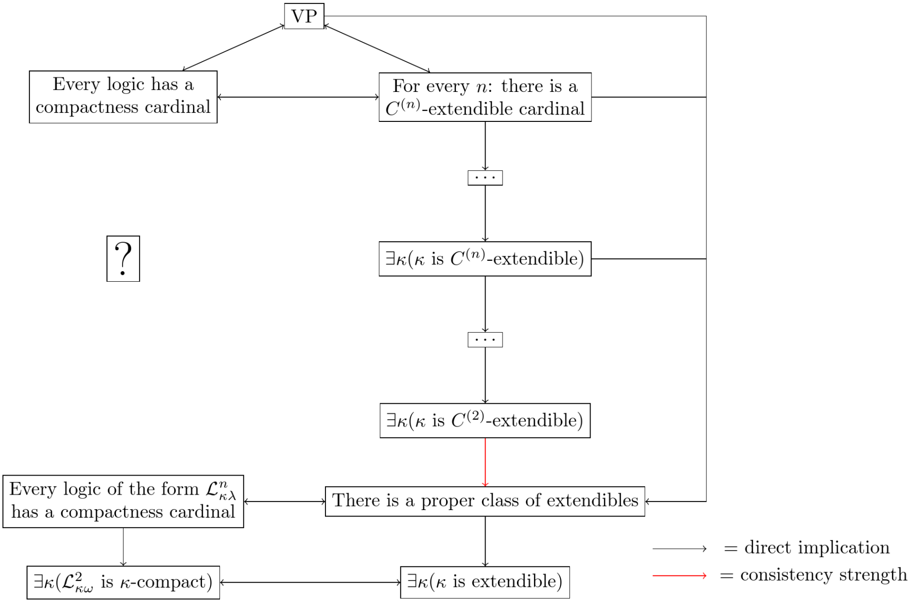
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Consider the following class:

$$C^{*(n)} := \{(M, E^M) \colon \exists \alpha \in C^{(n)} : (M, E^M) \cong (V_\alpha, \in)\}.$$

Let $Q_{C^{*(n)}}$ be the quantifier with the following semantics:

$(M, E^M) \models Q_{C^{*(n)}} xyz\varphi(x)\psi(y,z)$ iff
there is a structure $\mathcal{B} \in C^{*(n)}$ s.t. $B = \{a \in A \colon A \models \varphi(a)\}$
and $E^{\mathcal{B}} = \{(a,b) \in B^2 \colon \mathcal{A} \models \psi(a,b)\}$.

Definition

For every n and every cardinal κ, we let

$$\mathcal{L}_\kappa^{(n)} := \mathcal{L}^2_{\kappa\omega}(Q_{C^{*(1)}}, \ldots, Q_{C^{*(n)}}),$$

i.e., second-order logic with infinitary conjunctions and disjunctions of size $< \kappa$, extended by the quantifiers $Q_{C^{*(1)}}, \ldots, Q_{C^{*(n)}}$.

With those logics, the following holds:

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For every n and every cardinal κ:

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Proof.

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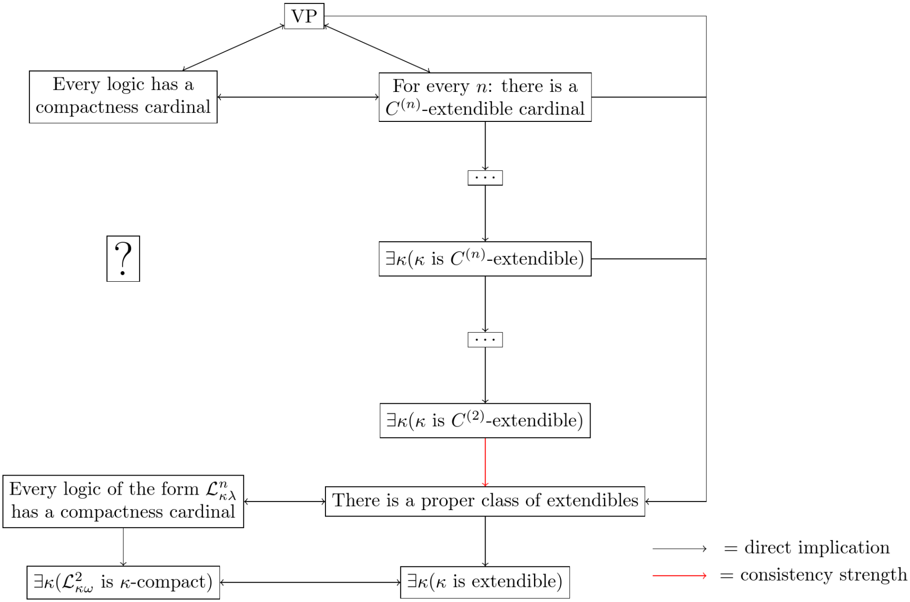
Will Boney similarly observed in [2] that κ is $C^{(n)}$-extendible iff the sort logic $\mathbb{L}_{\kappa\kappa}^{s,\Sigma_n}$ is κ-compact. I am not sure about the relation of these results. It is clear that $\mathcal{L}_\kappa^{(n)} \leq \mathbb{L}_{\kappa\kappa}^{s,\Sigma_{n+1}}$.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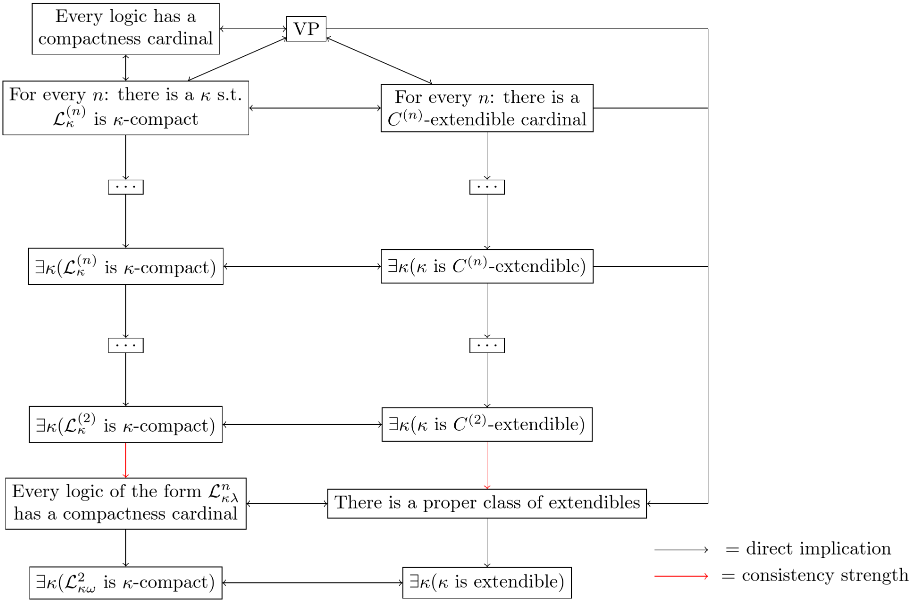
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Bibliography

[1] J. Bagaria. $C^{(n)}$-cardinals. In *Archive for Mathematical Logic*, 51(3-4):213-240, 2012.

[2] W. Boney. Model Theoretic Characterizations of Large Cardinals. In *Israel J. Math.*, 236:133-181, 2020.

[3] M. Magidor. On the Role of Supercompact and Extendible Cardinals in Logic. In *Israel J. Math.*, 10:147-157, 1971.

[4] J.A. Makowsky. Vopěnka's Principle and Compact Logics. In *J. Symb. L.*, 50(1):42-48, 1985.