Medvedev’s logic LM is not finitely axiomatizable

STiHAC

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Upper Bound of $ML_{ccc}$
### Upper Bound of $ML_{ccc}$

#### Definition

The *modal formulas* are built from propositional variables, Boolean connectives and $\Box$ ("necessary"), $\Diamond$ ("possible").

#### Definition

We call any directed graph $(K, E)$ a *Kripke frame*; 
If $v : Prop \to \mathcal{P}(K)$ is a function (called a valuation function), we shall call $(K, E, v)$ a *Kripke model*. 
For any Kripke model $(K, E, v)$ and $x \in K$, we define a *satisfaction relation* for modal formulas recursively as follows:

- $(K, E, v, x) \models p$ iff $x \in v(p)$;
- $(K, E, v, x) \models \phi \land \psi$ iff $(K, E, v, x) \models \phi$ and $(K, E, v, x) \models \psi$;
- $(K, E, v, x) \models \phi \lor \psi$ iff $(K, E, v, x) \models \phi$ or $(K, E, v, x) \models \psi$;
- $(K, E, v, x) \models \neg \phi$ iff $(K, E, v, x) \not\models \phi$;
- $(K, E, v, x) \models \Box \phi$ iff for all $y$ such that $xEy$, $(K, E, v, y) \models \phi$;
- $(K, E, v, x) \models \Diamond \phi$ iff there is a $y$ such that $xEy$ and $(K, E, v, y) \models \phi$. 


Upper Bound of $ML_{ccc}$

**Definition**

If $\phi$ is a modal formula, we say that it is valid in $(K, E, \nu)$ if $(K, E, \nu, x) \models \phi$ for every $x \in K$.

We say that it is valid in $(K, E)$ if it is valid in every Kripke model on $(K, E)$.

If $C$ is a class of Kripke frames, we write $ML(C)$ for the set of modal formulas valid in all frames $(K, E) \in C$.

**Definition**

A modal logic (in this talk) is a set $\lambda$ of modal formulas closed under substitution, modus ponens, and necessitation ($A \in \lambda$ only if $\Box A \in \lambda$), containing classical tautologies and axioms of $S4$:

$\Box (\phi \rightarrow \psi) \rightarrow (\Box \phi \rightarrow \Box \psi)$, $\neg \Diamond \phi \leftrightarrow \Box \neg \phi$, $\Box \phi \rightarrow \phi$, $\Box \phi \rightarrow \Box \Box \phi$. 
Upper Bound of $ML_{ccc}$

We take $M$ to be a countable transitive model of ZFC and $\mathcal{P}$ the class of ccc-partial orders, and interpret

$M \models \diamond \phi$ as a statement for "$\phi$ holds in some forcing extension of $M$ by forcing with a partial order in $\mathcal{P}$";

$M \models \Box \phi$ as a statement for "$\phi$ holds in all forcing extensions of $M$ by forcing with partial orders in $\mathcal{P}$".

**Definition**

For every model of set theory $M$, we can consider $(\text{Mult}_c(M), \leq_{ccc})$ as a Kripke frame. A valuation function $\nu_c : \text{Prop} \to \mathcal{P}(\text{Mult}_c(M))$ is called ccc forcing-set theoretic if there is an assignment $p \mapsto \sigma_p$ assigning a sentence in the language of set theory to any propositional variable in such a way that $\nu_c(p) = \{ N \in \text{Mult}_c(M) ; N \models \sigma_p \}$. We call a Kripke model $((\text{Mult}_c(M), \leq_{ccc}), \nu_c)$ ccc forcing-set theoretic if $\nu_c$ is a set theoretic valuation function.
Upper Bound of $ML_{ccc}$

<table>
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<th>Definition</th>
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<td>We can now define the modal logic of ccc forcing of $M$ by $ML_{ccc}(M) := {\Psi; \Psi \text{ is satisfied at } M \text{ in all ccc forcing-set theoretic Kripke models on } (\text{Mult}<em>c(M), \leq</em>{ccc})}$. The modal logic of ccc forcing is $ML_{ccc} := \bigcap{ML_{ccc}(M); M \models \text{ZFC is a countable model}}$.</td>
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### Theorem (Hamkins, Loewe)

If ZFC is consistent, then the ZFC-provable modal logic of ccc forcing, $ML_{ccc}$, is included in $S4tBA$.

### Theorem (Inamdar)

If ZFC is consistent, then the ZFC-provable modal logic of ccc forcing, $ML_{ccc}$, is included in $S4sBA$. 
Upper Bound of $ML_{ccc}$

**Definition**

The *finite partial function algebra* on $n$ elements is represented by the set $A_n$ of partial functions from $n$ to $\{S, F\}$ and $a < b \in A_n$ iff $a = b \upharpoonright dom(a)$.

**Definition**

The modal theory $\mathcal{FPFA}$ is defined to be modal assertions which are true in all Kripke models whose frame is a finite partial function algebra.
## Upper Bound of $ML_{ccc}$

<table>
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<td>If ZFC is consistent, then the ZFC-provable modal logic of ccc forcing, $ML_{ccc}$, is included in $\mathcal{FPFA}$.</td>
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Upper Bound of $ML_{ccc}$

**Theorem**

$\mathcal{FPA} \subset S4tBA \cap S4sBA.$
Upper Bound of $ML_{ccc}$

Is $FPFA$ finitely axiomatizable?
Is $FPFA$ the best upper bound of the ZFC-provable modal logic of ccc forcing?
Medvedev’s Logic
Medvedev’s Logic

Definition
An intuitionistic Kripke frame is a pair $\mathcal{F} = \langle W, \leq \rangle$ such that $W$ is a non-empty set and $\leq$ is a partial order, that is, a reflexive, transitive and anti-symmetric binary relation on $W$.
A valuation in a frame $\mathcal{F} = \langle W, \leq \rangle$ is a map $V$ associating with each propositional variable $p$ some subset $V(p)$ of $W$ such that, for every $x \in V(p)$ and $y \in W$, $x \leq y$ implies $y \in V(p)$.
An intuitionistic Kripke model is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is an intuitionistic Kripke frame and $V$ a valuation in $\mathcal{F}$.
**Definition**

Let $\mathcal{F} = \langle W, \leq \rangle$ be a model and $x$ a point in the frame $\mathcal{F} = \langle W, \leq \rangle$. We inductively define $x \models \phi$ as follows:

1. $(\mathcal{M}, x) \models p$ iff $x \in V(p)$;
2. $(\mathcal{M}, x) \models \phi \land \psi$ iff $(\mathcal{M}, x) \models \phi$ and $(\mathcal{M}, x) \models \psi$;
3. $(\mathcal{M}, x) \models \phi \lor \psi$ iff $(\mathcal{M}, x) \models \phi$ or $(\mathcal{M}, x) \models \psi$;
4. $(\mathcal{M}, x) \models \phi \rightarrow \psi$ iff for all $y$, $(x \leq y$ and $\mathcal{M}, y \models \phi$) implies $(\mathcal{M}, y) \models \psi$;
5. $(\mathcal{M}, x) \not\models \bot$. 


Definition

A formula $\phi$ is true in $\mathcal{M}$ if $(\mathcal{M}, x) \models \phi$ for every $x \in \mathcal{F}$; in this case we write $\mathcal{M} \models \phi$.

The formula $\phi$ is valid in the frame $\mathcal{F}$ if $\phi$ is true in all models on $\mathcal{F}$; in this case we write $\mathcal{F} \models \phi$.

If $S$ is an intermediate logic, a frame $\mathcal{F}$ is an $S$-frame if all formulas of $S$ are valid in $\mathcal{F}$.

Finally, we say that $\phi$ is valid in a class of Kripke frame $\mathcal{C}$, and write $\mathcal{C} \models \phi$, if $\mathcal{F} \models \phi$, for every $\mathcal{F} \in \mathcal{C}$. The logic $\text{Log}(\mathcal{C})$ is the set of formulas that are valid in $\mathcal{C}$.
\textbf{p-morphism}

A map $f$ from $\mathcal{F}$ to $\mathcal{F}'$ is a \textit{p-morphism} if
(1). for all $x, y \in \mathcal{F}$, $x \leq y$ implies $f(x) \leq f(y)$,
(2). for all $x \in \mathcal{F}$ and all $z \in \mathcal{F}'$, $f(x) \leq z$ implies that there exists a $y \in \mathcal{F}$ such that $x \leq y$ and $f(y) = z$.

In case $f$ is onto, we say that $\mathcal{F}'$ is a \textit{p-morphic image} of $\mathcal{F}$.

Recall that $p$-morphisms preserve validity. That is, if $f$ is a $p$-morphism from $\mathcal{F}$ onto $\mathcal{F}'$, then $\mathcal{F} \models \phi$ implies $\mathcal{F}' \models \phi$ for every formula $\phi$. 
Jankov-de Jongh Theorem
For every finite rooted frame $\mathcal{F}$, there is a formula $\chi(\mathcal{F})$ such that for every frame $\mathcal{G}$, $\mathcal{G} \not\models \chi(\mathcal{F})$ iff $\mathcal{F}$ is a $p$-morphically image of a generated subframe of $\mathcal{G}$.
The formula $\chi(\mathcal{F})$ is called the Jankov-de Jongh formula of $\mathcal{F}$.

Corollary
If $C$ is a class of finite Kripke frames closed under rooted generated subframes, then for every finite rooted frame $\mathcal{F}$, $\mathcal{F} \models \text{Log}(C)$ iff $\mathcal{F}$ is a $p$-morphically image of some frame in $C$. 
Definition (Maksimova, Shehtman and Skvorcov)

For a finite non-empty set $D$, let $P^0(D)$ denote the Kripke frame

$$P^0(D) = \langle \{X \subseteq D \mid X \neq \emptyset \}, \supseteq \rangle.$$ 

We call $P^0(D)$ a Medvedev’s frame. The intermediate logic $\textbf{LM}$ is the logic of all Medvedev frames, that is, the set of formulas that are valid in all Medvedev frames.

It is not hard to see that the class of Medvedev frames is closed under rooted generated subframes. Thus a frame is an $\textbf{LM}$-frame iff it is a $p$-morphic image of some Medvedev frame.
Definition (Chinese Lantern $CL(s, n)$)

For $n \geq 1$ and $s \geq 3$, the Chinese Lantern is the frame $CL(s, n)$ formed by the set: $\{(i, j) \in \omega \times \omega \mid (0 \leq i \leq s - 3, 0 \leq j \leq 1) \lor (i = s - 2, 0 \leq j \leq n - 1) \lor (i = s - 1, j = 0)\}$, with the accessibility relation being an ordering: $(i, j) \leq (i', j')$ iff $i > i' \lor (i, j) = (i', j')$.

For each natural number $k \geq 1$, let $G_k$ be the frame $CL(k + 3, 2^{k+3})$. 
Definition (Chinese Lantern $CL'(s, n, m)$)

For $m \leq s - 3$, the frame $CL'(s, n, m)$ formed by the set:
\[ \{(i, j) \in \omega \times \omega \mid (0 \leq i \leq s - 3, i \neq m, 0 \leq j \leq 1) \lor (i = m, j = 0) \lor (i = s - 2, 0 \leq j \leq n - 1) \lor (i = s - 1, j = 0)\}, \]
with the accessibility relation being an ordering: $(i, j) \leq (i', j')$ iff
\[ i > i' \lor (i, j) = (i', j'). \]

For each natural number $k \geq 1$ and each $m \leq k$, let $G_k^m$ be the frame $CL'(k + 3, 2^{k+3}, m)$. 
Proposition

For each natural number $k \geq 1$, the frame $G_k$ is not LM-frame.

Lemma

Let $D$ be a finite non-empty set and let $\mathcal{F}$ be a finite rooted frame. If $\mathcal{F}$ is a $p$-morphic image of some $P^0(D)$, then either $\mathcal{F}$ has some point with a single immediate successor or the branching degree of any $x$ in $\mathcal{F}$ is less than $2^{d(x)}$, where $d(x)$ is the depth of the subframe generated by $x$. 
**Proposition**
For each natural number $k \geq 1$ and each $m \leq k$, the frame $G_k^m$ is LM-frame.

**Lemma**
If $\mathcal{F}$ is a finite rooted frame with a greatest element, then $\mathcal{F}$ is a $p$-morphic image of some $P^0(D)$.

**Lemma**
If a finite rooted frame $\mathcal{F} = \langle W, R \rangle$ is a $p$-morphic image of some $P^0(D)$, then the frame $\mathcal{G} = \langle V, S \rangle$ defined by $V = W \cup \{a, b\}$ and $S = R \cup \{(x, a), (x, b); x \in W\}$ is a $p$-morphic image of some $P^0(D')$. 
Proposition

Let $\phi$ be a formula with $k$ variables. There exists a natural number $m \leq k$ such that $G_k \models \phi$ iff $G_k^m \models \phi$. 
Medvedev’s Logic

**Theorem (Maksimova, Shehtman and Skvorcov)**

Medvedev’s logic **LM** is not finitely axiomatizable.
Modal Counterparts
Modal Counterparts

Definition
Let $A$ be a propositional formula. The Tarski-translation of $A$ is defined recursively as follows:
\[
\begin{align*}
T(p_n) &= \Box p_n; \\
T(\neg A) &= \Box \neg T(A); \\
T(A \land B) &= T(A) \land T(B); \\
T(A \lor B) &= T(A) \lor T(B); \\
T(A \rightarrow B) &= \Box (T(A) \rightarrow T(B)).
\end{align*}
\]

The well-known translation takes every intuitionistic formula to a modal formula.
For a set $S$ of modal formulas we put $T^{-1}(S) = \{ A; T(A) \in S \}$. If $L$ is a modal logic, then $T^{-1}(L)$ is an intermediate logic, $L$ is called a 3counterpart of $T^{-1}(L)$.
Modal Counterparts

For a modal or an intermediate logic $L$, let $\epsilon(L)$ be the set of all its extensions, that is, of modal (respectively, intermediate) logics containing $L$. It is ordered by inclusion.

**Proposition**

Let $L = H + S$ be an intermediate logic. Then

1. $\tau(L) = S4 + T(S)$ is the least modal counterpart of $L$;
2. $L$ has the greatest modal counterpart (denoted by $\sigma(L)$);
3. $\sigma$ is an isomorphism between $\epsilon(H)$ and $\epsilon(Grz)$ (the Blok-ESakia isomorphism);
4. $\sigma(L) = Grz + T(S)$. 


Corollary

$L$ is finitely axiomatizable iff $\sigma(L)$ is.

Proof.

"Only if" is a consequence of $\sigma(L) = Grz + T(S)$.

On the other hand, if $L$ is not finitely axiomatizable, it is the union of an ascending chain of logics: $L_0 \subset L_1 \subset \ldots$, thus $\sigma(L)$ is the union of a chain: $\sigma(L_0) \subset \sigma(L_1) \subset \ldots$, so $\sigma(L)$ is not finitely axiomatizable. $$\square$$
Modal Counterparts

Proposition
Let $C$ be a class of finite partially ordered frames. Then $ML(C) = \sigma(\text{Log}(C))$.

Proof.
If $\mathcal{F} \in C$ and $A \in \text{Log}(C)$, then $\mathcal{F} \models A$, i.e. $\mathcal{F} \models T(A)$. Thus $\tau(\text{Log}(C)) \subseteq ML(C)$. Since the Grzegorczyk axiom is valid in any finite partially order frame, $\sigma(\text{Log}(C)) \subseteq ML(C)$. $T^{-1}(ML(\mathcal{F})) = \text{Log}(\mathcal{F})$ because of the definition of $\text{Log}(\mathcal{F})$. Hence $T^{-1}(ML(C)) = \text{Log}(C)$, and $ML(C) \subseteq \sigma(\text{Log}(C))$. \qed
Modal Counterparts

Conclusion

\[ \sigma(\text{LM}) = ML(\{P^0(D); D \text{ is finite and non-empty}\}) = S4tBA. \]

\[ S4tBA \text{ is not finitely axiomatizable since } \text{LM} \text{ is not.} \]
Thank you!
[1]. Valentin Shehtman, Modal Counterparts of Medvedev Logic of Finite Problems Are Not Finitely Axiomatizable
[2]. Gaelle Fontaine, Axiomatization of ML and Cheq
[3]. Han Xiao, Note on FPFA
[4]. L. Maksimova, V. Shehtman, and D. Skvorcov. The impossibility of a finite axiomatization of Medvedev’s logic of finitary problems