Medvedev's logic LM is not finitely axiomatizable

STiHAC

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Upper Bound of ML_{ccc}

Upper Bound of ML_{ccc}

Definition

The *modal formulas* are built from propositional variables, Boolean connectives and $\Box$ ("necessary"), $\Diamond$ ("possible").

Definition

We call any directed graph (K, E) a *Kripke frame*;
If $v : Prop \to \mathcal{P}(K)$ is a function (called a valuation function), we shall call (K, E, v) a *Kripke model*.
For any Kripke model (K, E, v) and $x \in K$, we define a *satisfaction relation* for modal formulas recursively as follows:
$(K, E, v, x) \models p$ iff $x \in v(p)$;
$(K, E, v, x) \models \phi \wedge \psi$ iff $(K, E, v, x) \models \phi$ and $(K, E, v, x) \models \psi$;
$(K, E, v, x) \models \phi \vee \psi$ iff $(K, E, v, x) \models \phi$ or $(K, E, v, x) \models \psi$;
$(K, E, v, x) \models \neg\phi$ iff $(K, E, v, x) \not\models \phi$;
$(K, E, v, x) \models \Box\phi$ iff for all y such that xEy, $(K, E, v, y) \models \phi$;
$(K, E, v, x) \models \Diamond\phi$ iff there is a y such that xEy and $(K, E, v, y) \models \phi$.

Upper Bound of ML_{ccc}

Definition

If ϕ is a modal formula, we say that it is valid in (K, E, v) if $(K, E, v, x) \models \phi$ for every $x \in K$.
We say that it is valid in (K, E) if it is valid in every Kripke model on (K, E).
If C is a class of Kripke frames, we write $ML(C)$ for the set of modal formulas valid in all frames $(K, E) \in C$.

Definition

A *modal logic* (in this talk) is a set λ of modal formulas closed under substitution, modus ponens, and necessitation ($A \in \lambda$ only if $\Box A \in \lambda$), containing classical tautologies and axioms of $S4$:
$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$, $\neg\Diamond\phi \leftrightarrow \Box\neg\phi$, $\Box\phi \to \phi$, $\Box\phi \to \Box\Box\phi$.

Upper Bound of ML_{ccc}

We take M to be a countable transitive model of ZFC and $\mathcal{P}$ the class of ccc-partial orders, and interpret
$M \models \Diamond\phi$ as a statement for "ϕ holds in some forcing extension of M by forcing with a partial order in $\mathcal{P}$";
$M \models \Box\phi$ as a statement for "ϕ holds in all forcing extensions of M by forcing with partial orders in $\mathcal{P}$".

Definition

For every model of set theory M, we can consider $(Mult_c(M), \leq_{ccc})$ as a Kripke frame. A valuation function $v_c : Prop \to \mathcal{P}(Mult_c(M))$ is called *ccc forcing-set theoretic* if there is an assignment $p \mapsto \sigma_p$ assigning a sentence in the language of set theory to any propositional variable in such a way that $v_c(p) = \{N \in Mult_c(M); N \models \sigma_p\}$. We call a Kripke model $((Mult_c(M), \leq_{ccc}), v_c)$ *ccc forcing-set theoretic* if v_c is a set theoretic valuation function.

Upper Bound of ML_{ccc}

Definition

We can now define the *modal logic of ccc forcing* of M by $ML_{ccc}(M) := \{\Psi; \Psi$ is satisfied at M in all *ccc forcing-set theoretic* Kripke models on $(Mult_c(M), \leq_{ccc})\}$.

The *modal logic of ccc forcing* is $ML_{ccc} := \bigcap\{ML_{ccc}(M)$; $M \models ZFC$ is a countable model $\}$.

Upper Bound of ML_{ccc}

Theorem (Hamkins, Loewe)

If ZFC is consistent, then the ZFC-provable modal logic of ccc forcing, ML_{ccc}, is included in $S4tBA$.

Theorem (Inamdar)

If ZFC is consistent, then the ZFC-provable modal logic of ccc forcing, ML_{ccc}, is included in $S4sBA$.

Upper Bound of ML_{ccc}

Definition

The *finite partial function algebra* on n elements is represented by the set A_n of partial functions from n to $\{S, F\}$ and $a < b \in A_n$ iff $a = b \restriction dom(a)$.

Definition

The modal theory $\mathcal{FPFA}$ is defined to be modal assertions which are true in all Kripke models whose frame is a finite partial function algebra.

Upper Bound of ML_{ccc}

Theorem

If ZFC is consistent, then the ZFC-provable modal logic of ccc forcing, ML_{ccc}, is included in $\mathcal{FPFA}$.

Upper Bound of ML_{ccc}

Theorem

$\mathcal{FPFA} \subset S4tBA \cap S4sBA$.

Upper Bound of ML_{ccc}

Is $\mathcal{FPFA}$ finitely axiomatizable?
Is $\mathcal{FPFA}$ the best upper bound of the ZFC-provable modal logic of ccc forcing?

Medvedev's Logic

Medvedev's Logic

Definition

An *intuitionistic Kripke frame* is a pair $\mathcal{F} = \langle W, \leq \rangle$ such that W is a non-empty set and $\leq$ is a partial order, that is, a reflexive, transitive and anti-symmetric binary relation on W.

A *valuation* in a frame $\mathcal{F} = \langle W, \leq \rangle$ is a map V associating with each propositional variable p some subset $V(p)$ of W such that, for every $x \in V(p)$ and $y \in W$, $x \leq y$ implies $y \in V(p)$.

An *intuitionistic Kripke model* is a pair $\mathcal{M} = \langle \mathcal{F}, V \rangle$, where $\mathcal{F}$ is an intuitionistic Kripke frame and V a valuation in $\mathcal{F}$.

Medvedev's Logic

Definition

Let $\mathcal{F} = \langle W, \leq \rangle$ be a model and x a point in the frame $\mathcal{F} = \langle W, \leq \rangle$. We inductively define $x \models \phi$ as follows:
$(\mathcal{M}, x) \models p$ iff $x \in V(p)$;
$(\mathcal{M}, x) \models \phi \wedge \psi$ iff $(\mathcal{M}, x) \models \phi$ and $(\mathcal{M}, x) \models \psi$;
$(\mathcal{M}, x) \models \phi \vee \psi$ iff $(\mathcal{M}, x) \models \phi$ or $(\mathcal{M}, x) \models \psi$;
$(\mathcal{M}, x) \models \phi \to \psi$ iff for all y, ($x \leq y$ and $\mathcal{M}, y \models \phi$) implies $(\mathcal{M}, y) \models \psi$;
$(\mathcal{M}, x) \not\models \perp$.

Medvedev's Logic

Definition

A formula ϕ is *true* in $\mathcal{M}$ if $(\mathcal{M}, x) \models \phi$ for every $x \in \mathcal{F}$; in this case we write $\mathcal{M} \models \phi$.
The formula ϕ is *valid* in the frame $\mathcal{F}$ if ϕ is true in all models on $\mathcal{F}$; in this case we write $\mathcal{F} \models \phi$.
If S is an intermediate logic, a frame $\mathcal{F}$ is an S-frame if all formulas of S are valid in $\mathcal{F}$.
Finally we say that ϕ is *valid* in a class of Kripke frame C, and write $C \models \phi$, if $\mathcal{F} \models \phi$, for every $\mathcal{F} \in C$. The logic $Log(C)$ is the set of formulas that are valid in C.

Medvedev's Logic

p-morphism

A map f from $\mathcal{F}$ to $\mathcal{F}'$ is a *p-morphism* if
(1). for all $x, y \in \mathcal{F}$, $x \leq y$ implies $f(x) \leq f(y)$,
(2). for all $x \in \mathcal{F}$ and all $z \in \mathcal{F}'$, $f(x) \leq z$ implies that there exists a $y \in \mathcal{F}$ such that $x \leq y$ and $f(y) = z$.

In case f is onto, we say that $\mathcal{F}'$ is a *p-morphic image* of $\mathcal{F}$.
Recall that p-morphisms preserve validity. That is, if f is a p-morphism from $\mathcal{F}$ onto $\mathcal{F}'$, then $\mathcal{F} \models \phi$ implies $\mathcal{F}' \models \phi$ for every formula ϕ.

Medvedev's Logic

Jankov-de Jongh Theorem

For every finite rooted frame $\mathcal{F}$, there is a formula $\chi(\mathcal{F})$ such that for every frame $\mathcal{G}$, $\mathcal{G} \not\models \chi(\mathcal{F})$ iff $\mathcal{F}$ is a p-morphic image of a generated subframe of $\mathcal{G}$.

The formula $\chi(\mathcal{F})$ is called the Jankov-de Jongh formula of $\mathcal{F}$.

Corollary

If C is a class of finite Kripke frames closed under rooted generated subframes, then for every finite rooted frame $\mathcal{F}$, $\mathcal{F} \models Log(C)$ iff $\mathcal{F}$ is a p-morphic image of some frame in C.

Medvedev's Logic

Definition (Maksimova, Shehtman and Skvorcov)

For a finite non-empty set D, let $P^0(D)$ denote the Kripke frame $P^0(D) = \langle \{X \subseteq D | X \neq \emptyset\}, \supseteq \rangle$.
We call $P^0(D)$ a $Medvedev's\ frame$. The intermediate logic **LM** is the logic of all Medvedev frames, that is, the set of formulas that are valid in all Medvedev frames.

It is not hard to see that the class of Medvedev frames is closed under rooted generated subframes.
Thus a frame is an **LM**-frame iff it is a p-morphic image of some Medvedev frame.

Medvedev's Logic

Definition (Chinese Lantern $CL(s, n)$)

For $n \geq 1$ and $s \geq 3$, the *Chinese Lantern* is the frame $CL(s, n)$ formed by the set: $\{(i, j) \in \omega \times \omega \mid (0 \leq i \leq s-3, 0 \leq j \leq 1) \bigvee (i = s-2, 0 \leq j \leq n-1) \bigvee (i = s-1, j = 0)\}$,
with the accessibility relation being an ordering: $(i, j) \leq (i', j')$ iff $i > i' \vee (i, j) = (i', j')$.

For each natural number $k \geq 1$, let G_k be the frame $CL(k+3, 2^{k+3})$.

Medvedev's Logic

Definition (Chinese Lantern $CL'(s, n, m)$)

For $m \leq s - 3$, the frame $CL'(s, n, m)$ formed by the set:
$\{(i, j) \in \omega \times \omega \mid (0 \leq i \leq s - 3, i \neq m, 0 \leq j \leq 1) \bigvee (i = m, j = 0) \bigvee (i = s - 2, 0 \leq j \leq n - 1) \bigvee (i = s - 1, j = 0)\}$,
with the accessibility relation being an ordering: $(i, j) \leq (i', j')$ iff $i > i' \vee (i, j) = (i', j')$.

For each natural number $k \geq 1$ and each $m \leq k$, let G_k^m be the frame $CL'(k + 3, 2^{k+3}, m)$.

Medvedev's Logic

Proposition

For each natural number $k \geq 1$, the frame G_k is not **LM**-frame.

Lemma

Let D be a finite non-empty set and let $\mathcal{F}$ be a finite rooted frame. If $\mathcal{F}$ is a p-morphic image of some $P^0(D)$, then either $\mathcal{F}$ has some point with a single immediate successor or the branching degree of any x in $\mathcal{F}$ is less than $2^{d(x)}$, where $d(x)$ is the depth of the subframe generated by x.

Medvedev's Logic

Proposition

For each natural number $k \geq 1$ and each $m \leq k$, the frame G_k^m is **LM**-frame.

Lemma

If $\mathcal{F}$ is a finite rooted frame with a greatest element, then $\mathcal{F}$ is a *p*-morphic image of some $P^0(D)$.

Lemma

If a finite rooted frame $\mathcal{F} = \langle W, R\rangle$ is a *p*-morphic image of some $P^0(D)$, then the frame $\mathcal{G} = \langle V, S\rangle$ defined by $V = W \cup \{a, b\}$ and $S = R \cup \{(x, a), (x, b); x \in W\}$ is a *p*-morphic image of some $P^0(D')$.

Proposition

Let ϕ be a formula with k variables. There exists a natural number $m \leq k$ such that $G_k \models \phi$ iff $G_k^m \models \phi$.

Medvedev's Logic

Theorem (Maksimova, Shehtman and Skvorcov)

Medvedev's logic **LM** is not finitely axiomatizable.

Modal Counterparts

Modal Counterparts

Definition

Let A be a propositional formula. The *Tarski-translation* of A is defined recursively as follows:
$T(p_n) = \Box p_n$;
$T(\neg A) = \Box\neg T(A)$;
$T(A \wedge B) = T(A) \wedge T(B)$;
$T(A \vee B) = T(A) \vee T(B)$;
$T(A \to B) = \Box(T(A) \to T(B))$.

The well-known translation takes every intuitionistic formula to a modal formula.
For a set S of modal formulas we put $T^{-1}(S) = \{A; T(A) \in S\}$. If L is a modal logic, then $T^{-1}(L)$ is an intermediate logic, L is called a 3*counterpart* of $T^{-1}(L)$.

Modal Counterparts

For a modal or an intermediate logic L, let $\epsilon(L)$ be the set of all its extensions, that is, of modal (respectively, intermediate) logics containing L. It is ordered by inclusion.

Proposition

Let $L = H + S$ be an intermediate logic. Then
(1) $\tau(L) = S4 + T(S)$ is the least modal counterpart of L;
(2) L has the greatest modal counterpart (denoted by $\sigma(L)$);
(3) σ is an isomorphism between $\epsilon(H)$ and $\epsilon(Grz)$(the *Blok-Esakia* isomorphism);
(4) $\sigma(L) = Grz + T(S)$.

Modal Counterparts

Corollary

L is finitely axiomatizable iff $\sigma(L)$ is.

Proof.

"Only if" is a consequence of $\sigma(L) = Grz + T(S)$.
On the other hand, if L is not finitely axiomatizable, it is the union of an ascending chain of logics: $L_0 \subset L_1 \subset \ldots$, thus $\sigma(L)$ is the union of a chain: $\sigma(L_0) \subset \sigma(L_1) \subset \ldots$, so $\sigma(L)$ is not finitely axiomtizable. $\square$

Modal Counterparts

Proposition

Let C be a class of finite partially ordered frames. Then $ML(C) = \sigma(Log(C))$.

Proof.

If $\mathcal{F} \in C$ and $A \in Log(C)$, then $\mathcal{F} \models A$, i.e. $\mathcal{F} \models T(A)$. Thus $\tau(Log(C)) \subseteq ML(C)$. Since the Grzegorczyk axiom is valid in any finite partially order frame, $\sigma(Log(C)) \subseteq ML(C)$.
$T^{-1}(ML(\mathcal{F})) = Log(\mathcal{F})$ because of the definition of $Log(\mathcal{F})$. Hence $T^{-1}(ML(C)) = Log(C)$, and $ML(C) \subseteq \sigma(Log(C))$. $\square$

Modal Counterparts

Conclusion

$\sigma(\mathbf{LM}) = ML(\{P^0(D); D \text{ is finite and non-empty}\}) = S4tBA$.
$S4tBA$ is not finitely axiomatizable since **LM** is not.

Thank you!

Reference

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