Set Theories through Ordinary Mathematics

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Contents

Idea

1. Stipulate some basic mathematical object, O
2. Stipulate some believable(!) endogenous axioms, A, for O
3. Stipulate an interpretation for the primitive notions of set theory, $\mathcal{I}$, in the relevant language.
4. See which set theory the structure $\langle O, A, \mathcal{I} \rangle$ models

Philosophical motivation: a candidate foundation $\langle O, A, \mathcal{I} \rangle$ must preserve the direction of plausibility - not all theorems as axioms!

Contents

Topos Model

Theorem (Mac Lane & Moerdijk [6] §10, Lawvere [5])

A well-pointed topos with a natural number object and choice models Bounded ZFC - Replacement.

Topos Model

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A well-pointed topos with a natural number object and choice models Bounded ZFC - Replacement.

Definition

Well pointed topos is a category which:

1. has finite limits
2. is Cartesian closed (internalises homomorphisms)
3. has a subobject classifier (identifies characteristic functions)
4. 1 is not initial (non-degenerate)
5. for $f, g : A \rightrightarrows B$, $f = g$ iff $fx = gx$ for every global element x of A (somewhat like extensionality).

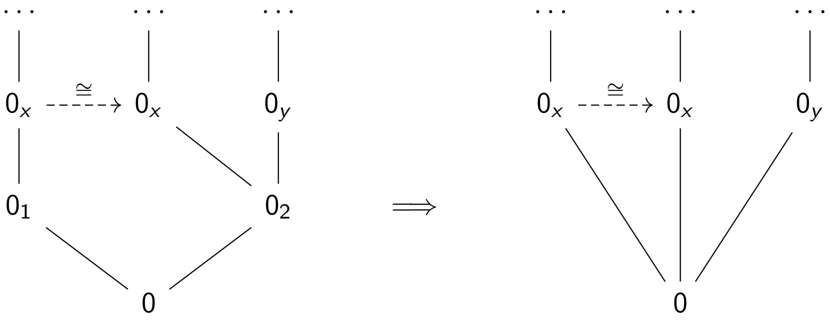
$\mathbb{C}$ has Choice if every epi splits, i.e. if $e : X \to Y$ is epi, then there is a morphism $s : Y \to X$ such that $e \circ s = id_Y$.

Topos Model - Simulating the Graph Model

Theorem (Mac Lane & Moerdijk [6] §10, Lawvere [5])

A well-pointed topos with a natural number object and choice models Bounded ZFC - Replacement.

Union: adjoin the representatives 0_1 and 0_2 at a root using colimits, then colimits again to quotient



Topos Model - Infinity

Theorem (Mac Lane & Moerdijk [6] §10, Lawvere [5])

A well-pointed topos with a natural number object (NNO) and choice models Bounded ZFC - Replacement.

Definition (NNO)

A NNO on a topos $\mathcal{E}$ is an object N of $\mathcal{E}$ with arrows

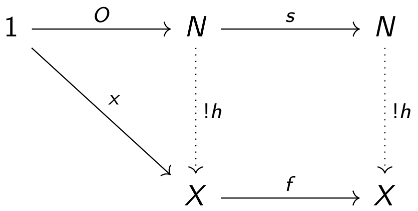
$$1 \xrightarrow{O} N \xrightarrow{s} N$$

such that for any object X of $\mathcal{E}$ with arrows x and f such that

$$1 \xrightarrow{x} X \xrightarrow{f} X$$

then there exists a unique $h : N \to N$ such that the following commute

Topos Model - Infinity



Category-ese for s is a successor function on N.

Topos Model - Parasitism?

Claim

A well-pointed topos has independent motivation as a foundational category.

Topos Model - Parasitism?

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A well-pointed topos has independent motivation as a foundational category.

Parasitical Claim

NNO and Choice have no independent motivation, besides modeling ω and the Axiom of Choice

So too for Replacement? The constraint on the category depends on "how much Replacement you want" ([7] 2.3.10).

Topos Model - Parasitism?

Does the parasitical claim hold up?

topos axiom	set axiom
NNO	Inf
Choice	AC
various replacement analogues	various strengths of Replacement

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Definition of a WPT

requirement	internalisation	set axiom
finite limits	products	Union (with Powers)
Cartesian closed	homomorphisms	-
subobject classifier	χ_f	Powers
1 is not initial	non-degeneracy	(Found?)
$f = g$ iff $\forall x$ global $fx = gx$	equality	Ext(+)

Topos Model

Parasitic or not, certain toposes can interpret standard set theories, Z, FinSet, ZC, etc.

Method: imitate the graph theoretic model and identify any "copies".

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Question

How about natural topos axioms to models strengthening of ZFC? Reflection principles?

Contents

Algebraic Model

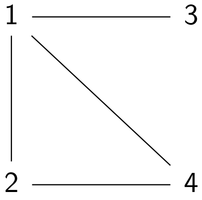
Question

Which basic algebraic entities and constructions are required for a model of a standard set theory?

Or for a substantial fragment of concrete mathematics

Encode a Graph

Natural approach: encode graphs again. E.g. G:



Adjacency matrix?

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

Encoding Finite Graphs

For this kind of representation, our foundation must include:

1. A collection, S, $|S| \geq 2$ (e.g. C_2)
2. The general theory of (2 dimensional) matrices on a collection S, $M^{\mathbf{On} \times \mathbf{On}}(S)$.

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No clear way to 'connect' the matrix-representations of graphs (i.e. coproducts/sums).

Link '3' of one copy of G to '4' of another copy to make an 8×8 matrix? More generally, we must define addition on arbitrary matrices in $M^{\mathbf{On}\times\mathbf{On}}(S)$. Unclear how to describe such an addition algebraically.

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Algebraic categories obviously have products.
They can have direct sums(/colimits), e.g. in $\mathrm{G{\scriptsize RP}}$, colimits are quotients of the free product by suitable congruences.

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Why algebraic interpretations don't work

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Which basic algebraic entities and constructions are required for a model of a standard set theory?

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Problem

No internal way to take direct sums(/colimits) of algebraic categories.

Using congruence and quotients relies on structure beyond the relevant algebraic theory.

Major restriction on expressiveness: set theory is closed under limits ($\sim$products) *and* colimits ($\sim$sums and unions). This seems an unavoidable problem of algebraic foundations.

Contents

A diversion into Positive Set Theory

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- Some PSTs are okay for constructions, e.g. have ordinals ([3]:1.3)
- in all known models, the sets are classes closed under a topology (κ-compact κ-topological T_2 spaces homeomorphic to their own hyperspace)
- One PST, Topological Set Theory has axioms like
  - If $A \subseteq \mathbb{T}$ is nonempty, then $\bigcap A$ is $\mathbb{T}$-closed.
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  - If a and b are $\mathbb{T}$-closed, then $a \cup b$ is $\mathbb{T}$-closed.
- but a strange family: no singletons, universal set, non-well-founded, only positive separation!
- very strong: GPK^+_ω has consistency strength proper class ordinal weakly compact [2]

Cheating topologically

Suppose $\mathrm{C} = \mathrm{TOP}$. Let M be a set-model of set theory T.[1] Then discrete spaces $M \times M, 2 \in \mathrm{C}$, and there is a TOP-map

$$\in_M : M \times M \to 2$$

Encode $\in$-relation as $\in_M^{-1}(\{1\})$. Then $\langle M \times M, \in_M^{-1}(\{1\}), \mathcal{I} \rangle$ models T.

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Problem

$\in$ is non-constructive, showing only that there is a model for ZFC 'somewhere in' TOP. Relies on prior knowledge of TOP.

Instead, we stipulate some basic entities and constructions, and *build* a category which contains a model of ZFC.

[1] Let T be no stronger than e.g. NBG with Choice

Cheating less badly

1. C is closed under
   - sums
   - quotients
   - finite products
2. $\omega + 1 \in \mathrm{C}$
3. $\forall \kappa \in Card^{\geq\omega}$, $\exists X \in \mathrm{C}$, $\exists U \subseteq X$ open discrete subset with $|X| = \kappa$ which witnesses tightness κ exactly.

Theorem (Dow & Watson [1])

If 1., 2., and 3. hold, then $\mathrm{C} = \mathrm{TOP}$.

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Theorem (Dow & Watson [1])

If 1., 2., and 3. hold, then $\mathrm{C} = \mathrm{TOP}$.

Problem

This essentially requires an ambient (external) set theory, especially for quotients.

Candidate Topological Model: First Axioms

More constructively...

topological axiom	set structure
0 space	Emptyset
1 space	singleton
$0 \neq 1$	non-degeneracy (Found?)
finite limits	finite products
finite colimits	finite unions
suitably full	"correct" sums and products
Sierpiński space	open and closed sets
ω	ω (Inf)
(countable) powers + discretisation	large sets (up to $\beth_\omega$)

Axiom

There is a 0 *space in* C, *i.e.* $\exists 0 \in \mathrm{C}$, *such that for any space* $x \in \mathrm{C}$, *there is a unique function* $! : 0 \to x$.

Axiom

There is a 1-point space in C, *i.e. unique function* $! : x \to 1$.

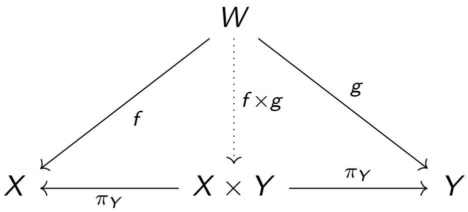
Axiom

$0 \neq 1$.

Axiom

C *has finite limits*

TOP has finite (co-)limits so this is reasonable. Note the binary product, $Z = X \times Y$:



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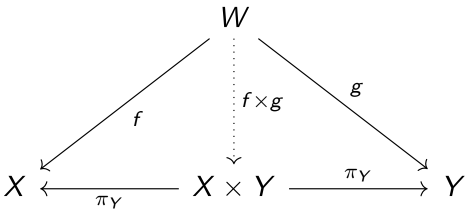


Axiom

C *has finite colimits.*

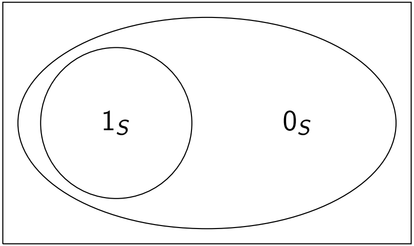
Proposition

If C *has enough morphisms,* $2 \in c$

Sierpiński Space, Opens

Axiom

The Sierpiński space is in C



Proposition

If C *has a enough morphisms* C *can interpret open sets, and closed sets internally.*

Countable products would be useful, e.g. for 2^ω.

Problem

Our logic is finitary, so no countable limits in the language.

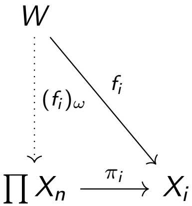
Problem

How to express countable families?

Things get strange: Countable Powers

Axiom

Suppose X_n are (somehow!) indexed by ω. Then $\exists Z = \prod_{n \in \omega} X_n \in \mathrm{C}$ i.e. for countably many maps $f_i : W \to X_i$, there is a unique $(f_i)_\omega : W \to \prod X_i$ such that all(!) diagrams of this shape commute:



Corollary (Cantor set)

$\bigcup_{i \in \omega} 2_{(i)} =: 2^\omega \in \mathrm{C}$

Other axioms

topological axiom	set structure
indiscretisation	arbitrary subsets
discretisation	big sets, compare cardinalities
topological separation	Sep
hyperspaces	[subobject classifier]
G_δ	?
unit interval	?
Stone-Čech compactificiations	?
$\beta\omega$	?
internal function space	2^ω (from 1, ω, finite limits)[2]
TOP congruences are C congruences	quotienting (e.g. $[0,1] = 2^\omega / R$)

[2] i.e. *without* countable products

Contents

Selection of References

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