Descriptive Set Theory with Absolutely No Choice Whatsoever

Lucas Wansner (UHH), Ned Wontner (ILLC, UvA)

25th February 2021
STiHAC-Forschungsseminar Mathematische Logik, UHH

Contents

Borel Sets

Definition (Borel sets)

The *Borel algebra* is defined as the smallest σ-algebra containing all open sets. We denote it by $\mathcal{B}$ and call its elements *Borel sets*.

Borel Sets

Definition (Borel sets)

The *Borel algebra* is defined as the smallest σ-algebra containing all open sets. We denote it by $\mathcal{B}$ and call its elements *Borel sets*.

Definition (Borel Hierarchy)

- $\mathbf{\Sigma}^0_1 := \{O \subseteq \omega^\omega : O \text{ is open}\}$,
- $\mathbf{\Pi}^0_\xi := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \mathbf{\Sigma}^0_\xi\}$,
- $\mathbf{\Sigma}^0_{\xi+1} := \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_\xi\}$,
- $\mathbf{\Sigma}^0_\lambda := \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\xi_n} \text{ and } \xi_n < \lambda\}$, for λ a limit, and
- $\mathbf{\Delta}^0_\xi := \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$.

Borel Sets

Definition (Borel sets)

The *Borel algebra* is defined as the smallest σ-algebra containing all open sets. We denote it by $\mathcal{B}$ and call its elements *Borel sets*.

Definition (Borel Hierarchy)

- $\mathbf{\Sigma}^0_1 := \{O \subseteq \omega^\omega : O \text{ is open}\}$,
- $\mathbf{\Pi}^0_\xi := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \mathbf{\Sigma}^0_\xi\}$,
- $\mathbf{\Sigma}^0_{\xi+1} := \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_\xi\}$,
- $\mathbf{\Sigma}^0_\lambda := \{\bigcup_{n\in\omega} A_n : A_n \in \mathbf{\Pi}^0_{\xi_n}$ and $\xi_n < \lambda\}$, for λ a limit, and
- $\mathbf{\Delta}^0_\xi := \mathbf{\Sigma}^0_\xi \cap \mathbf{\Pi}^0_\xi$.

A set B is Borel if and only if there is a ξ such that $B \in \mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$.

Definition (well-founded tree)

- A *tree* is a non-empty subset $T \subseteq \omega^{<\omega}$ which is closed under initial segments.

Trees

Definition (well-founded tree)

- A *tree* is a non-empty subset $T \subseteq \omega^{<\omega}$ which is closed under initial segments.
- Let $s, t \in T$. We say s is a *successor of* t *in* T if there is a $k \in \omega$ such that $s = t^\frown k$ and denote the set of all successors of t in T by $\mathrm{Succ}_T(t)$.

Trees

Definition (well-founded tree)

- A *tree* is a non-empty subset $T \subseteq \omega^{<\omega}$ which is closed under initial segments.
- Let $s, t \in T$. We say s is a *successor of* t *in* T if there is a $k \in \omega$ such that $s = t^\frown k$ and denote the set of all successors of t in T by $\mathrm{Succ}_T(t)$.
- Let $s, t \in \omega^{<\omega}$. We define $s \leq t$ if there is a $k \in \omega$ such that $s \restriction k = t$.

Trees

Definition (well-founded tree)

- A *tree* is a non-empty subset $T \subseteq \omega^{<\omega}$ which is closed under initial segments.
- Let $s, t \in T$. We say s is a *successor of* t *in* T if there is a $k \in \omega$ such that $s = t^\frown k$ and denote the set of all successors of t in T by $\mathrm{Succ}_T(t)$.
- Let $s, t \in \omega^{<\omega}$. We define $s \leq t$ if there is a $k \in \omega$ such that $s \restriction k = t$.
- A tree T *well-founded* if $(T, \leq \cap T^2)$ is well-founded.

Trees

Definition (well-founded tree)

- A *tree* is a non-empty subset $T \subseteq \omega^{<\omega}$ which is closed under initial segments.
- Let $s, t \in T$. We say s is a *successor of t in T* if there is a $k \in \omega$ such that $s = t^\frown k$ and denote the set of all successors of t in T by $\mathrm{Succ}_T(t)$.
- Let $s, t \in \omega^{<\omega}$. We define $s \leq t$ if there is a $k \in \omega$ such that $s \restriction k = t$.
- A tree T *well-founded* if $(T, \leq \cap T^2)$ is well-founded.

Remark

Let $\pi : \omega \to \omega^{<\omega}$ be a bijection and let T be a tree. We define $c \in 2^\omega$ by $c(k) = 1$ if and only if $\pi(k) \in T$. Then c is a code for T. We shall often identify a tree with its code.

Borel Codes and Codable Borel Sets

Definition (Borel code)

We fix a bijection $\pi : \omega \to \omega^{<\omega}$. A *Borel code* is a real $c \in 2^\omega$ which codes a well-founded tree T_c. We define recursively for every note t of T_c:

$$B_t := \begin{cases} \emptyset & \text{if } \mathrm{Succ}_{T_c}(t) = \emptyset \wedge t = \emptyset, \\ [\pi(k)] & \text{if } \mathrm{Succ}_{T_c}(t) = \emptyset \wedge t(\mathrm{lh}(t) - 1) = k, \\ \omega^\omega \setminus B_s & \text{if } \mathrm{Succ}_{T_c}(t) = \{s\}, \\ \bigcup_{s \in \mathrm{Succ}_{T_c}(t)} B_s & \text{otherwise.} \end{cases}$$

We set $B_c := B_\emptyset$.

Borel Codes and Codable Borel Sets

Definition (Borel code)

We fix a bijection $\pi : \omega \to \omega^{<\omega}$. A *Borel code* is a real $c \in 2^\omega$ which codes a well-founded tree T_c. We define recursively for every note t of T_c:

$$B_t := \begin{cases} \emptyset & \text{if } \mathrm{Succ}_{T_c}(t) = \emptyset \wedge t = \emptyset, \\ [\pi(k)] & \text{if } \mathrm{Succ}_{T_c}(t) = \emptyset \wedge t(\mathrm{lh}(t) - 1) = k, \\ \omega^\omega \setminus B_s & \text{if } \mathrm{Succ}_{T_c}(t) = \{s\}, \\ \bigcup_{s \in \mathrm{Succ}_{T_c}(t)} B_s & \text{otherwise.} \end{cases}$$

We set $B_c := B_\emptyset$.

Definition (codable Borel set)

A set B is *codable Borel* if there is a Borel code c such that $B_c = B$. We denote the set of all codable Borel sets by $\mathcal{B}^*$.

Borel vs. Codable Borel in ZFC

Theorem ($AC_\omega(\omega^\omega)$)

A set of reals is Borel if and only if it is codable Borel.

Borel vs. Codable Borel in ZFC

Theorem ($AC_\omega(\omega^\omega)$)

A set of reals is Borel if and only if it is codable Borel.

Proof.

It is clear that every codable Borel set is Borel. We prove that every set $B \in \mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$ is codable Borel by induction:

□

Borel vs. Codable Borel in ZFC

Theorem ($AC_\omega(\omega^\omega)$)

A set of reals is Borel if and only if it is codable Borel.

Proof.

It is clear that every codable Borel set is Borel. We prove that every set $B \in \mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$ is codable Borel by induction:

- $B \in \mathbf{\Sigma}^0_1$: Let $T := \{\emptyset\} \cup \{\langle k \rangle : \pi(k) \subseteq B\}$. Then T is a Borel code for B.

□

Borel vs. Codable Borel in ZFC

Theorem ($AC_\omega(\omega^\omega)$)

A set of reals is Borel if and only if it is codable Borel.

Proof.

It is clear that every codable Borel set is Borel. We prove that every set $B \in \mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$ is codable Borel by induction:

- $B \in \mathbf{\Sigma}^0_1$: Let $T := \{\emptyset\} \cup \{\langle k \rangle : \pi(k) \subseteq B\}$. Then T is a Borel code for B.
- $B \in \mathbf{\Pi}^0_\xi$: Let T be a Borel code for $\omega^\omega \setminus B$. Then $T' := \{\langle 0 \rangle^\frown t : t \in T\}$ is a Borel code for B.

□

Borel vs. Codable Borel in ZFC

Theorem ($AC_\omega(\omega^\omega)$)

A set of reals is Borel if and only if it is codable Borel.

Proof.

It is clear that every codable Borel set is Borel. We prove that every set $B \in \mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$ is codable Borel by induction:

- $B \in \mathbf{\Sigma}^0_1$: Let $T := \{\emptyset\} \cup \{\langle k \rangle : \pi(k) \subseteq B\}$. Then T is a Borel code for B.
- $B \in \mathbf{\Pi}^0_\xi$: Let T be a Borel code for $\omega^\omega \setminus B$. Then $T' := \{\langle 0 \rangle ^\frown t : t \in T\}$ is a Borel code for B.
- $B \in \mathbf{\Sigma}^0_{\xi+1}$: Then there are $B_k \in \mathbf{\Pi}^0_\xi$ such that $B = \bigcup_{k \in \omega} B_k$. **Choose** Borel codes T_k for B_k. Without loss of generality, $T_k \neq \{\emptyset\}$. Then $T := \{\langle k \rangle ^\frown t : t \in T_k\}$ is a Borel code for B.

□

Borel vs. Codable Borel in ZFC

Theorem ($AC_\omega(\omega^\omega)$)

A set of reals is Borel if and only if it is codable Borel.

Proof.

It is clear that every codable Borel set is Borel. We prove that every set $B \in \mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi$ is codable Borel by induction:

- $B \in \mathbf{\Sigma}^0_1$: Let $T := \{\emptyset\} \cup \{\langle k \rangle : \pi(k) \subseteq B\}$. Then T is a Borel code for B.
- $B \in \mathbf{\Pi}^0_\xi$: Let T be a Borel code for $\omega^\omega \setminus B$. Then $T' := \{\langle 0 \rangle^\frown t : t \in T\}$ is a Borel code for B.
- $B \in \mathbf{\Sigma}^0_{\xi+1}$: Then there are $B_k \in \mathbf{\Pi}^0_\xi$ such that $B = \bigcup_{k \in \omega} B_k$. **Choose** Borel codes T_k for B_k. Without loss of generality, $T_k \neq \{\emptyset\}$. Then $T := \{\langle k \rangle^\frown t : t \in T_k\}$ is a Borel code for B.
- $B \in \mathbf{\Sigma}^0_\lambda$: Analogous to $B \in \mathbf{\Sigma}^0_{\xi+1}$. $\square$

Feferman-Lévy Model and Symmetric Submodels

Theorem (Feferman-Lévy)

There is a model of ZF *in which the reals are a countable union of countable sets and* ω_1 *is singular.*

Feferman-Lévy Model and Symmetric Submodels

Theorem (Feferman-Lévy)

There is a model of ZF *in which the reals are a countable union of countable sets and* ω_1 *is singular.*

Proof idea

We start with L and take a symmetric submodel using a forcing notion which Lévy collapses all ω_n to ω.

Feferman-Lévy Model and Symmetric Submodels

Theorem (Feferman-Lévy)

There is a model of ZF *in which the reals are a countable union of countable sets and* ω_1 *is singular.*

Proof idea

We start with L and take a symmetric submodel using a forcing notion which Lévy collapses all ω_n to ω.

Remark

In the Feferman-Lévy model every set of reals is $\mathbf{\Delta}^0_4$. In particular, every set of reals is Borel.

Borel vs. Codable Borel in ZF

Theorem

ZF $\nvdash \mathcal{B} = \mathcal{B}^*$.

Borel vs. Codable Borel in ZF

Theorem

ZF $\nvdash \mathcal{B} = \mathcal{B}^*$.

Proof.

Each codeable Borel set is a coded by a real number, so there is a surjection $f : \omega^\omega \twoheadrightarrow \mathcal{B}^*$. Meanwhile there are ZF models, e.g. the Feferman-Lévy model, where $\mathcal{P}(\omega^\omega) = \mathcal{B}$, so by Cantor's Theorem, $\mathcal{B} \neq \mathcal{B}^*$ in this models. □

Borel vs. Codable Borel in ZF

Theorem

ZF $\nvdash \mathcal{B} = \mathcal{B}^*$.

Proof.

Each codeable Borel set is a coded by a real number, so there is a surjection $f : \omega^\omega \twoheadrightarrow \mathcal{B}^*$. Meanwhile there are ZF models, e.g. the Feferman-Lévy model, where $\mathcal{P}(\omega^\omega) = \mathcal{B}$, so by Cantor's Theorem, $\mathcal{B} \neq \mathcal{B}^*$ in this models. □

Proposition

Every $\mathbf{\Sigma}^0_2$ and every $\mathbf{\Pi}^0_2$ set of reals is codable Borel.

Borel Hierarchy

Facts

Let ξ be an ordinal.

- The Borel hierarchy is increasing, i.e. $\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$.

Borel Hierarchy

Facts

Let ξ be an ordinal.

- The Borel hierarchy is increasing, i.e. $\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$.
- $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, and $\mathbf{\Delta}^0_\xi$ are closed under continuous preimages.

Borel Hierarchy

Facts

Let ξ be an ordinal.

- The Borel hierarchy is increasing, i.e. $\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$.
- $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, and $\mathbf{\Delta}^0_\xi$ are closed under continuous preimages.

Proposition

ZF $\nvdash$ $\mathbf{\Sigma}^0_2$ is not closed under countable unions.

Borel Hierarchy

Facts

Let ξ be an ordinal.

- The Borel hierarchy is increasing, i.e. $\mathbf{\Sigma}^0_\xi \cup \mathbf{\Pi}^0_\xi \subseteq \mathbf{\Delta}^0_{\xi+1}$.
- $\mathbf{\Sigma}^0_\xi$, $\mathbf{\Pi}^0_\xi$, and $\mathbf{\Delta}^0_\xi$ are closed under continuous preimages.

Proposition

ZF $\nvdash$ $\mathbf{\Sigma}^0_2$ is not closed under countable unions.

Proof.

We suppose for a contradiction, that $\mathbf{\Sigma}^0_2$ is closed under countable unions. Then in Feferman-Lévy model every set of reals is $\mathbf{\Sigma}^0_2$ and so every set of reals is codable Borel. But this is a contradiction. □

Length of the Borel Hierarchy

Definition (Length of Borel hierarchy)

We call the least ordinal ξ such that $\mathbf{\Sigma}^0_\xi = \mathbf{\Pi}^0_\xi$ the *length of the Borel hierarchy*.

Length of the Borel Hierarchy

Definition (Length of Borel hierarchy)

We call the least ordinal ξ such that $\mathbf{\Sigma}^0_\xi = \mathbf{\Pi}^0_\xi$ the *length of the Borel hierarchy*.

Theorem ($AC_\omega(\omega^\omega)$)

The length of the Borel hierarchy is ω_1.

Length of the Borel Hierarchy

Definition (Length of Borel hierarchy)

We call the least ordinal ξ such that $\mathbf{\Sigma}^0_\xi = \mathbf{\Pi}^0_\xi$ the *length of the Borel hierarchy*.

Theorem ($AC_\omega(\omega^\omega)$)

The length of the Borel hierarchy is ω_1.

Remark

- The theorem is not provable in ZF, e.g. in the Feferman-Lévy model the length is 4.

Length of the Borel Hierarchy

Definition (Length of Borel hierarchy)

We call the least ordinal ξ such that $\mathbf{\Sigma}^0_\xi = \mathbf{\Pi}^0_\xi$ the *length of the Borel hierarchy*.

Theorem ($AC_\omega(\omega^\omega)$)

The length of the Borel hierarchy is ω_1.

Remark

- The theorem is not provable in ZF, e.g. in the Feferman-Lévy model the length is 4.
- It is provable in ZF that $\mathbf{\Sigma}^0_3 \neq \mathbf{\Pi}^0_3$. Therefore, 4 is the least possible length.

Length of the Borel Hierarchy

Definition (Length of Borel hierarchy)

We call the least ordinal ξ such that $\mathbf{\Sigma}^0_\xi = \mathbf{\Pi}^0_\xi$ the *length of the Borel hierarchy*.

Theorem ($AC_\omega(\omega^\omega)$)

The length of the Borel hierarchy is ω_1.

Remark

- The theorem is not provable in ZF, e.g. in the Feferman-Lévy model the length is 4.
- It is provable in ZF that $\mathbf{\Sigma}^0_3 \neq \mathbf{\Pi}^0_3$. Therefore, 4 is the least possible length.
- Every codable Borel set is $\mathbf{\Delta}^0_{\omega_1}$.

Length of the Borel Hierarchy

The standard proof of the theorem only uses the fact that ω_1 is regular. We can use it to prove a weakening of the theorem in ZF.

Length of the Borel Hierarchy

The standard proof of the theorem only uses the fact that ω_1 is regular. We can use it to prove a weakening of the theorem in ZF.

Theorem

Let λ be a limit ordinal with $\mathrm{cof}(\lambda) > \omega$. Then

1. $\mathbf{\Sigma}^0_\lambda = \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi$ *and*
2. *the length of the Borel hierarchy is less or equal to λ.*

Length of the Borel Hierarchy

The standard proof of the theorem only uses the fact that ω_1 is regular. We can use it to prove a weakening of the theorem in ZF.

Theorem

Let λ be a limit ordinal with $\operatorname{cof}(\lambda) > \omega$. Then

1. $\mathbf{\Sigma}^0_\lambda = \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi$ *and*
2. *the length of the Borel hierarchy is less or equal to λ.*

Proof.

1. By definition, $\bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi \subseteq \mathbf{\Sigma}^0_\lambda$. Let $B \in \mathbf{\Sigma}^0_\lambda$. Then there are $\xi_n < \lambda$ and $B_n \in \mathbf{\Pi}^0_{\xi_n}$ such that $B = \bigcup_n B_n$. Let $\xi := \lim_n \xi_n$. Since $\operatorname{cof}(\lambda) > \omega$, $\xi < \lambda$. Then $B_n \in \mathbf{\Pi}^0_\xi$ and so $B \in \mathbf{\Sigma}^0_{\xi+1}$. Since λ is a limit, $\xi + 1 < \lambda$. □

Length of the Borel Hierarchy

The standard proof of the theorem only uses the fact that ω_1 is regular. We can use it to prove a weakening of the theorem in ZF.

Theorem

Let λ be a limit ordinal with $\mathrm{cof}(\lambda) > \omega$. Then

1. $\mathbf{\Sigma}^0_\lambda = \bigcup_{\xi<\lambda} \mathbf{\Sigma}^0_\xi$ *and*
2. *the length of the Borel hierarchy is less or equal to λ.*

Proof.

2. It is enough to show that $\mathbf{\Delta}^0_\lambda$ is a σ-algebra. By definition, it is closed under complements. We only have to check that it is closed under countable unions. Let $B_n \in \mathbf{\Delta}^0_\lambda$, let $\xi_n < \lambda$ be minimal such that $B_n \in \mathbf{\Pi}^0_{\xi_n}$, and let $\xi := \lim_n \xi_n$. Since $\mathrm{cof}(\lambda) > \omega$, $\xi < \lambda$. Then $B := \bigcup_n B_n \in \mathbf{\Sigma}^0_{\xi+1}$ and so $B \in \mathbf{\Delta}^0_{\xi+2}$. Since λ is a limit, $\xi + 2 < \lambda$ and so $B \in \mathbf{\Delta}^0_\lambda$. □

Length of the Borel Hierarchy

Theorem (Miller)

There is a model of ZF *such that the length of the Borel hierarchy is* ω_2.

Length of the Borel Hierarchy

Theorem (Miller)

There is a model of ZF *such that the length of the Borel hierarchy is* ω_2*.*

Theorem (Miller)

For every limit ordinal α *such that* $\omega \leq \alpha < \omega_2^V$*, there is a model of* ZF *such that the length of the Borel hierarchy is* α*.*

Length of the Borel Hierarchy

Theorem (Miller)

There is a model of ZF *such that the length of the Borel hierarchy is* ω_2.

Theorem (Miller)

For every limit ordinal α *such that* $\omega \leq \alpha < \omega_2^V$, *there is a model of* ZF *such that the length of the Borel hierarchy is* α.

Theorem (Miller)

Suppose V *is a countable transitive model of* ZF *in which every* ω_α *has countable cofinality. Then for every ordinal* λ *in* V, *there model of* ZF *with the same* ω_α*'s as* V *and the length of the Borel hierarchy is greater* λ.

Contents

Analytic Sets

Definition (Analytic set)

A set of reals is *analytic* if either it is empty or it is a continuous image of ω^ω.

Analytic Sets

Definition (Analytic set)

A set of reals is *analytic* if either it is empty or it is a continuous image of ω^ω.

Proposition

Let A be a set of reals. The following are equivalent:

1. A is analytic,
2. A is the continuous image of a codable Borel set, and
3. A is the projection of a codable Borel set.

Analytic Sets

Definition (Analytic set)

A set of reals is *analytic* if either it is empty or it is a continuous image of ω^ω.

Proposition

Let A be a set of reals. The following are equivalent:

1. A is analytic,
2. A is the continuous image of a codable Borel set, and
3. A is the projection of a codable Borel set.

Remark

Every codable Borel set is analytic. But ZF does not prove that every Borel set is analytic. Otherwise, in the Feferman-Lévy we would get a surjection from the reals on its own power set.

Projective Hierarchy

Definition (Projective Hierarchy)

- $\mathbf{\Sigma}^1_1 := \{A \subseteq \omega^\omega : A \text{ is analytic}\}$,
- $\mathbf{\Pi}^1_n := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \mathbf{\Sigma}^1_n\}$,
- $\mathbf{\Sigma}^1_{n+1} := \{A \subseteq \omega^\omega : A \text{ is the projection of an } A' \in \mathbf{\Pi}^1_n\}$, and
- $\mathbf{\Delta}^1_n := \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

Projective Hierarchy

Definition (Projective Hierarchy)

- $\mathbf{\Sigma}^1_1 := \{A \subseteq \omega^\omega : A \text{ is analytic}\}$,
- $\mathbf{\Pi}^1_n := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \mathbf{\Sigma}^1_n\}$,
- $\mathbf{\Sigma}^1_{n+1} := \{A \subseteq \omega^\omega : A \text{ is the projection of an } A' \in \mathbf{\Pi}^1_n\}$, and
- $\mathbf{\Delta}^1_n := \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

Facts

- The projective Hierarchy is increasing, i.e. $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$.

Projective Hierarchy

Definition (Projective Hierarchy)

- $\mathbf{\Sigma}^1_1 := \{A \subseteq \omega^\omega : A \text{ is analytic}\}$,
- $\mathbf{\Pi}^1_n := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \mathbf{\Sigma}^1_n\}$,
- $\mathbf{\Sigma}^1_{n+1} := \{A \subseteq \omega^\omega : A \text{ is the projection of an } A' \in \mathbf{\Pi}^1_n\}$, and
- $\mathbf{\Delta}^1_n := \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

Facts

- The projective Hierarchy is increasing, i.e. $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$.
- $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ are closed under continuous preimages.

Projective Hierarchy

Definition (Projective Hierarchy)

- $\mathbf{\Sigma}^1_1 := \{A \subseteq \omega^\omega : A \text{ is analytic}\}$,
- $\mathbf{\Pi}^1_n := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \mathbf{\Sigma}^1_n\}$,
- $\mathbf{\Sigma}^1_{n+1} := \{A \subseteq \omega^\omega : A \text{ is the projection of an } A' \in \mathbf{\Pi}^1_n\}$, and
- $\mathbf{\Delta}^1_n := \mathbf{\Sigma}^1_n \cap \mathbf{\Pi}^1_n$.

Facts

- The projective Hierarchy is increasing, i.e. $\mathbf{\Sigma}^1_n \cup \mathbf{\Pi}^1_n \subseteq \mathbf{\Delta}^1_{n+1}$.
- $\mathbf{\Sigma}^1_n$, $\mathbf{\Pi}^1_n$, and $\mathbf{\Delta}^1_n$ are closed under continuous preimages.
- $\mathbf{\Sigma}^1_n$ is closed under projections.

Projective Hierarchy

Definition (Projective Hierarchy)

- $\boldsymbol{\Sigma}^1_1 := \{A \subseteq \omega^\omega : A \text{ is analytic}\}$,
- $\boldsymbol{\Pi}^1_n := \{A \subseteq \omega^\omega : \omega^\omega \setminus A \in \boldsymbol{\Sigma}^1_n\}$,
- $\boldsymbol{\Sigma}^1_{n+1} := \{A \subseteq \omega^\omega : A \text{ is the projection of an } A' \in \boldsymbol{\Pi}^1_n\}$, and
- $\boldsymbol{\Delta}^1_n := \boldsymbol{\Sigma}^1_n \cap \boldsymbol{\Pi}^1_n$.

Facts

- The projective Hierarchy is increasing, i.e. $\boldsymbol{\Sigma}^1_n \cup \boldsymbol{\Pi}^1_n \subseteq \boldsymbol{\Delta}^1_{n+1}$.
- $\boldsymbol{\Sigma}^1_n$, $\boldsymbol{\Pi}^1_n$, and $\boldsymbol{\Delta}^1_n$ are closed under continuous preimages.
- $\boldsymbol{\Sigma}^1_n$ is closed under projections.
- $\boldsymbol{\Sigma}^1_n \neq \boldsymbol{\Pi}^1_n$ for every $n \in \omega$.

Projective Hierarchy

Theorem (Suslin, $AC_\omega(\omega^\omega)$)

$\mathcal{B} = \boldsymbol{\Delta}^1_1$.

Projective Hierarchy

Theorem (Suslin, $AC_\omega(\omega^\omega)$)

$\mathcal{B} = \mathbf{\Delta}^1_1$.

Lemma (Lusin, $AC_\omega(\omega^\omega)$)

For every disjoint analytic sets A, A' there is a Borel set B such that $A \subseteq B$ and A' is disjoint from B.

Projective Hierarchy

Theorem (Suslin, $AC_\omega(\omega^\omega)$)

$\mathcal{B} = \mathbf{\Delta}^1_1$.

Lemma

For every disjoint analytic sets A, A' there is a **codable Borel** *set B such that $A \subseteq B$ and A' is disjoint from B.*

Projective Hierarchy

Theorem (Suslin, $AC_\omega(\omega^\omega)$)

$\mathcal{B} = \mathbf{\Delta}^1_1$.

Lemma

For every disjoint analytic sets A, A' there is a **codable Borel** *set B such that $A \subseteq B$ and A' is disjoint from B.*

Corollary

$\mathcal{B}^* = \mathbf{\Delta}^1_1 \subseteq \mathcal{B}$.

Projective Hierarchy

Theorem (Suslin, $AC_\omega(\omega^\omega)$)

$\mathcal{B} = \mathbf{\Delta}^1_1$.

Lemma

For every disjoint analytic sets A, A' there is a **codable Borel** *set B such that $A \subseteq B$ and A' is disjoint from B.*

Corollary

$\mathcal{B}^* = \mathbf{\Delta}^1_1 \subseteq \mathcal{B}$.

Remark

ZF does not prove that $\mathbf{\Delta}^1_1$ is a σ-algebra.

$\mathcal{B} = \mathcal{B}^*$

Proposition

The following are equivalent:

1. $\mathcal{B} = \mathcal{B}^*$,
2. $\mathbf{\Delta}^1_1$ is a σ-algebra,
3. $\mathcal{B} \subseteq \mathbf{\Delta}^1_1$, and
4. $\mathcal{B} \subseteq \mathbf{\Sigma}^1_1$.

$\mathcal{B} = \mathcal{B}^*$

Proposition

The following are equivalent:

1. $\mathcal{B} = \mathcal{B}^*$,
2. $\mathbf{\Delta}^1_1$ is a σ-algebra,
3. $\mathcal{B} \subseteq \mathbf{\Delta}^1_1$, and
4. $\mathcal{B} \subseteq \mathbf{\Sigma}^1_1$.

Theorem (Ikegami-Schlicht)

$AC_\omega(\mathbf{\Pi}^1_1) \rightarrow \mathcal{B} = \mathcal{B}^* \rightarrow AC_\omega(\mathcal{B}) \Rightarrow \omega_1$ *is regular.*

Contents

Restricted Choice Principles

Definition

Let Γ be a pointclass. We denote the statement "for every sequence $\langle A_k : k \in \omega \rangle$ of non-empty sets in Γ, there is a sequence $\langle a_k : k \in \omega \rangle$ of real numbers such that $a_k \in A_k$ for every $k \in \omega$" by $AC_\omega(\Gamma)$.

Restricted Choice Principles

Definition

Let Γ be a pointclass. We denote the statement "for every sequence $\langle A_k : k \in \omega \rangle$ of non-empty sets in Γ, there is a sequence $\langle a_k : k \in \omega \rangle$ of real numbers such that $a_k \in A_k$ for every $k \in \omega$" by $AC_\omega(\Gamma)$.

Proposition

ZF $\vdash AC_\omega(\mathbf{\Sigma}^0_1) + AC_\omega(\mathbf{\Pi}^0_1)$.

Restricted Choice Principles

Definition

Let Γ be a pointclass. We denote the statement "for every sequence $\langle A_k : k \in \omega \rangle$ of non-empty sets in Γ, there is a sequence $\langle a_k : k \in \omega \rangle$ of real numbers such that $a_k \in A_k$ for every $k \in \omega$" by $AC_\omega(\Gamma)$.

Proposition

ZF $\vdash AC_\omega(\mathbf{\Sigma}^0_1) + AC_\omega(\mathbf{\Pi}^0_1)$.

Remark

ZF also proves $AC_\omega(\mathbf{\Delta}^0_2)$, but ZF does not prove $AC_\omega(\mathbf{\Sigma}^0_2)$ or $AC_\omega(\mathbf{\Pi}^0_2)$.

Uniform Choice Principles

Definition (Kanovei)

Let Γ be a pointclass. We denote the statement "for every set $A \subseteq \omega \times \omega^\omega$ with $A \in \Gamma$ and domain ω there is a sequence $\langle a_k : k \in \omega \rangle$ such that $(k, a_k) \in A$ for every $k \in \omega$" by $AC^U_\omega(\Gamma)$.

Uniform Choice Principles

Definition (Kanovei)

Let Γ be a pointclass. We denote the statement "for every set $A \subseteq \omega \times \omega^\omega$ with $A \in \Gamma$ and domain ω there is a sequence $\langle a_k : k \in \omega \rangle$ such that $(k, a_k) \in A$ for every $k \in \omega$" by $AC^U_\omega(\Gamma)$.

Theorem (Kanovei)

1. $AC^U_\omega(\mathbf{\Pi}^1_n) \Leftrightarrow AC^U_\omega(\mathbf{\Sigma}^1_{n+1})$.
2. ZF $\vdash AC^U_\omega(\mathbf{\Pi}^1_1)$.
3. $AC^U_\omega(\mathbf{\Sigma}^1_n) \not\Rightarrow AC^U_\omega(\mathbf{\Sigma}^1_{n+1})$

Uniform vs Non-Uniform

Proposition

Let Γ be a pointclass that is closed under continuous preimages. Then $AC_\omega(\Gamma)$ implies $AC_\omega^U(\Gamma)$.

Uniform vs Non-Uniform

Proposition

Let Γ be a pointclass that is closed under continuous preimages. Then $AC_\omega(\Gamma)$ implies $AC_\omega^U(\Gamma)$.

Proof.

Let $A \subseteq \omega \times \omega^\omega$ be in Γ with domain ω. Since Γ is closed under continuous preimages, every $A_k := \{a : \langle k \rangle ^\frown a \in A\}$ is in Γ. By $AC_\omega(\Gamma)$, there is sequence $\langle a_k : k \in \omega \rangle$ such that $a_k \in A_k$ for every $k \in \omega$. Then $(k, a_k) \in A$ for every $k \in \omega$. □

Uniform vs Non-Uniform

Proposition

Let Γ be a pointclass that is closed under continuous preimages, countable unions, and products with closed sets. Then $AC_\omega(\Gamma)$ and $AC_\omega^U(\Gamma)$ are equivalent.

Uniform vs Non-Uniform

Proposition

Let Γ be a pointclass that is closed under continuous preimages, countable unions, and products with closed sets. Then $AC_\omega(\Gamma)$ and $AC_\omega^U(\Gamma)$ are equivalent.

Proof.

We only have to show that $AC_\omega^K(\Gamma)$ implies $AC_\omega(\Gamma)$. Let $\langle A_k : k \in \omega \rangle$ be a sequence of non-empty sets in Γ. Then $A := \bigcup_{k \in \omega} \{k\} \times A_k$ is in Γ. By $AC_\omega^U(\Gamma)$, there is a sequence $\langle a_k : k \in \omega \rangle$ such that $(k, a_k) \in A$ for every $k \in \omega$. Then $a_k \in A_k$ for every $k \in \omega$. □

Diagram

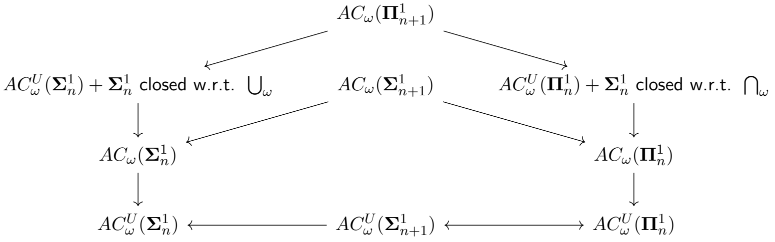
Lemma

$AC_\omega(\mathbf{\Pi}^1_{n+1})$ implies $\mathbf{\Sigma}^1_n$ is closed under countable unions and intersections.

Diagram

Lemma

$AC_\omega(\mathbf{\Pi}^1_{n+1})$ implies $\mathbf{\Sigma}^1_n$ is closed under countable unions and intersections.



Thank You!

Selection of References

[1] Fremlin D. H. *Measure theory*, volume 5. Torres Fremlin, 2000.

[2] Ikegami D. and Schlicht P. *Forcing and generic absolutness without choice.* preprint, 2021.

[3] Jech, T. J. *Set theory*, the third millennium edition, revised and expanded., Springer Monographs in Mathematics, Springer-Verlag, Berlin, 2003.

[4] Jech T. J. *The axiom of choice*, North-Holland Publishing Co., Amsterdam, 1973.

[5] Kanovei V. *On descriptive forms of the countable axiom of choice* Investigations on nonclassical logics and set theory, Work Collect, 3–136, 1979.

[6] Miller A. W. *Long borel hierarchies.*, Mathematical Logic Quarterly, 54(3):307–322, 2008.