Let $I$ be a proper $\sigma$-ideal and let $Q$ be a forcing notion. We say

- $Q$ is a forcing notion for $P_I$ if for every $R \in P_I$ and every $\omega$-generic filter $G$, there is a $C \in P_I$ such that $C \subseteq G$ and $\forall X \subseteq G \; X \in P_I$.

- $Q$ is an extension for $P_I$ for every forcing model $M \models \text{ZFC}^+$ and $N = \mathbb{V}[G]$, $x$ is $P_I$-generic over $V$.

Let $\mathbb{L}(G_{\omega_1})$ be a forcing notion satisfying the conditions above.

Start with $\mathbb{L}$ and we perform an iteration of $\mathbb{L}(G_{\omega_1})$. The resulting model $\mathbb{L}(G_{\omega_1})[G]$ will satisfy the desired properties.
Let $I$ be a $\mathcal{P}_1$-ideal s.t. Negari's Theorem holds and let $Q$ be a quasi-continuous $\mathcal{P}_1$-ideal.

Start with $I$ and put $\mathcal{P}_1$-ideals $Q$ with $Q$ with countable support.

Let $Q$ be a $\mathcal{P}_1$-generic filter over $\mathcal{L}_\mathcal{P}_1$. Now let

$$\forall B \in \mathcal{P}_1^\omega \exists C \subseteq B \exists x \in x \text{ is not } I\text{-quasi-generic over } \mathcal{L}_\mathcal{P}_1.$$
Example

\[ \forall \mathcal{T} \exists \mathcal{F} \forall x \exists y \] 

1. \[ \forall \mathcal{T} \exists \mathcal{F} : T \text{ forced tree, } \mathcal{M}(T) \models \mathcal{F} \] 

Claim: \[ \mathcal{F} \] is an answer for \[ \forall \mathcal{F} \]

Proof: \[ \bigcup \mathcal{N} \in \mathcal{N}(\mathcal{F}) \]

Let \[ \mathcal{N}_x = \{ \mathcal{T} \in \mathcal{F} : T \text{ forced tree, } \mathcal{M}(T) \models \mathcal{F} \} \]. For every \[ \frac{1}{2} \leq \varepsilon < 1 \], \[ \mathcal{N} \] and \[ \mathcal{N}_x \] are forcing equivalent (Trends in connections between answers and games).

Let \( \mathcal{F}_x \) be on \( \mathcal{N}_x \)-generic \( \mathcal{T} \) and \( \mathcal{P}_x := \bigcup \mathcal{N}_x \). Then \( \mathcal{P}_x \) is a closed set with \( \mu(\mathcal{P}_x) = \varepsilon \). Let \( \mathcal{N} \) be a \( \mathcal{N}(\mathcal{F}) \)-generic \( \mathcal{T} \) coded in \( \mathcal{N}_x \) and let \( \mathcal{T} \in \mathcal{N}_x \). Then there is \( \exists \mathcal{T}_0 \in \mathcal{N}_x \) \( \mathcal{T} \in \mathcal{N}_x \) and \( \exists \mathcal{F}_0 \in \mathcal{N}_x \) such that \( \mathcal{T} \in \mathcal{N}_x \) is closed.

\[ \Rightarrow \mathcal{P}_x \cap \mathcal{N} = \emptyset \]

\[ \Rightarrow \forall \mathcal{N} \in \mathcal{N}_x \quad \mathcal{N} \cap \mathcal{P}_x = \emptyset \]

\[ \Rightarrow \forall \mathcal{N} \in \mathcal{N}_x \quad \mathcal{N} \cup \mathcal{P}_x = \mathcal{C} \quad \forall \mathcal{N} \in \mathcal{N}_x \]

\[ \Rightarrow \forall \mathcal{N} \in \mathcal{N}_x \quad \mathcal{N} \cup \mathcal{P}_x = \mathcal{C} \quad \forall \mathcal{N} \in \mathcal{N}_x \]

Let \( \mathcal{B} \in \mathcal{P}_x \) \( \mathcal{N} \) is \( \mathcal{B} \)-good for \( \mathcal{F} \) \( \Rightarrow \mathcal{B} \setminus ( ) \leq \mathcal{B} \)
2) \( \Omega \times \Omega \) is an arrow for \( C \)

\[ U \Omega := \{ (\alpha, D) : \alpha \in \Omega_{\omega_0}, \text{D is open dense in } 2^\omega \} \]

\[ \{ x \in \omega_1 : \exists \omega \leq x \} \]

Claim: \( U \Omega \) is an arrow for \( C \)

Proof: Let \( \zeta \) be a \( \Omega \)-generic filter and let \( X = (\alpha, D) \). Then \( x \in (\zeta_{\omega_1}) \)

We consider \( \bigcap_{\omega \in \omega_1} O_{\omega} X(\omega) = \emptyset \)

Claim: \( E \) is convex and for every \( x \in U \Omega \), \( R_x \cap E = \emptyset \)

Proof (1st): \( U \Omega \), \( O_{\omega} X(\omega) \) is open dense for every \( \omega \in \omega_1 \)

Proof: Let \( s \in \omega_0 \) and \( D^s := \{ (\alpha, D) : \exists k \geq s, 3t+s \notin (\alpha) \} \)

Proof: Let \( g \in \omega_0 \) and \( D^g := \{ (\alpha, D) : \exists k \geq g, 3t+s \notin (\alpha) \} \)

Proof: Let \( (\alpha, D) \in U \Omega \), since \( D \) is open dense, there is \( s, t \geq s \) s.t. \( \varnothing_s \leq D \).

Let \( (\alpha', D') \in U \Omega \). Since \( D' \) is open dense, there is \( s, t \geq s \) s.t. \( \varnothing'_s \leq D' \).

Claim: \( D \cap D' \neq \emptyset \).

Proof: \( (\alpha, D) \in U \Omega \) and \( (\alpha', D') \in U \Omega \).

\( \forall s \in \omega_0 \exists t \geq s \varnothing \in U \Omega \).

\( \forall s \in \omega_0 \exists t \geq s \varnothing \in U \Omega \).
$\Rightarrow E$ is connected.

Let $N$ be non-empty and closed in $U$. Then $D_u = \{(u, D) : D \cap N = \emptyset \}$ is closed.

Hence, there is a $(v, D) \in D_u$ s.t. $U \cap E = \emptyset$. Then for any $x \in D \cap U \cap E$,

$D_x(k) \subseteq D$, hence $D_x(k) \cap N = \emptyset$ and so $E$ is disjoint from $N$.

\[ R := \{ P \subseteq 2^{\omega} : P \text{ is pruned tree and } \mu([\mathcal{P}]) = \frac{1}{2} \} \]

Let $\pi : 2^{\omega} \to 2^{\omega}$ be the canonical bijection:

\[ \pi(P)(n) := \begin{cases} 1 & \pi^{-1}(n) \in P \\ 0 & \text{otherwise} \end{cases} \]

$\forall P \in \mathcal{C}^\omega$, we can show that $\pi \in \mathcal{R}$ is Grt. $\overline{E}$

Let $A := \{ T : \pi \in \mathcal{K} \text{ region } \mathcal{R} \}$

Every region is closed. But not every closed is a region.

Every region is not $E$.
A \in R \text{ is } \varepsilon\text{-regular if } \forall T \in A \exists \varepsilon T \in A \text{ s.t. } (S) \setminus A = (S) \setminus (A \cup \varepsilon T)

Note: The \varepsilon\text{-regular sets form a \sigma-algebra containing all Borel sets in } R.

A \in R \text{ is } \varepsilon\text{-regular iff its } \mathcal{I}_\varepsilon\text{-regular}

\begin{align*}
\forall x \in \mathbb{R}, \exists \omega \in \mathbb{N} \setminus \{0\}, \forall \varepsilon \in (0, 1), \exists \delta \in (0, 1) \quad &\varepsilon \in B_{1/2} \\
\forall (\sigma, D) \in \mathcal{L} \quad &\langle \sigma, D \rangle := \{ x | \sigma x \in A \text{ and } \text{diam}(\sigma x) \leq D \} \\
\langle \sigma, D \rangle \leq \langle \sigma, \varepsilon \rangle \quad &\text{iff } \langle \sigma, D \rangle \leq \langle \sigma, \varepsilon \rangle
\end{align*}