Let I be a proper σ-ideal and let $\mathbb{Q}$ be a forcing notion. We say

- $\mathbb{Q}$ is a quasi-amoeba for $\mathbb{P}_I$ if for every $B \in \mathbb{P}_I$ and every $\mathbb{Q}$-generic filter G, there is a $\boxed{C \in \mathbb{P}_I^{V[G]}}$ such that $C \subseteq B$ and, $V[G] \models \forall x \in C$ x is $\mathbb{P}_I$-generic over V.

$\leq$

$$\mathbb{P}_I := \mathcal{B}(\omega^\omega) \setminus I$$
$$= \{X \in BC : B_X \notin I\}$$

$V = L[G_{\lambda_0}]$

W $V[G]$

$\forall B \, \exists C \in \mathbb{P}_I$

$L[G_{\lambda_0+1}] =$

- $\mathbb{Q}$ is an amoeba for $\mathbb{P}_I$ $\ldots$

for every larger model $W \supseteq V[G]$ $\quad W \models \forall x \in C$ x is $\mathbb{P}_I$-generic over V

$\supseteq$

$L[G_{\lambda_0}][G]$

T is a sacks tree only containing sacks-reals

S_1, S_2 | Start with L and we perform a ω_2 iteration of

Let I be a σ-ideal s.t Ikegami's Theorem holds and let $\mathbb{Q}$ be a quasi-generic for $\mathbb{P}_I$ and suppose that $\mathbb{Q}$ is proper.

Start with L and perform a ω_1-iteration $\mathbb{P}_{\omega_1}$ with $\mathbb{Q}$ with countable support.

Let G be a $\mathbb{P}_{\omega_1}$-generic filter over L. ~~and let~~ ~~$\forall B \in \mathbb{P}^{L[G]}$~~

$\forall B \in \mathbb{P}_I^{L[G]}\ \exists C \leq B \quad C \cap \{x : x \text{ ist not } I\text{-quasi-generic over } L[r]\}$

Let $B \in \mathbb{P}_I^{L[G]}$, $r \in \omega^\omega$, ~~let~~ Since the iteration is proper, there is some $\alpha_0 < \omega_1$ s.t. $B, r \in L[G_{\alpha_0}]$. In $L[G_{\alpha_0+1}]$ there is some $C \leq B$ s.t. for every $x \in C$ x is $\mathbb{P}_I$-generic over $L[G_{\alpha_0}]$

x is I-quasi-generic over $L[G_{\alpha_0}]$
$L[r]$

Π^1_2

$L[G] \models \exists C \leq B\ \forall x \in C\ I\text{-quasi-generic over } L[r]$

$\Rightarrow$ the set of non I-quasi-generics is I-null $\overset{\text{Ikegami}}{\Rightarrow} \Sigma^1_2(I)$

Δ^1_3

Example

$\forall t\in T\ \exists s\in T\ s\supseteq t.$

1.) $\mathbb{A} := \{T \subseteq 2^{<\omega} : T \text{ pruned tree}, \mu([T]) \geq \frac{1}{2}\}$ $\quad T\leq S \Leftrightarrow T \subseteq S$

Claim $\mathbb{A}$ is an amoeba for $\mathbb{B}$

Proof $\bigcup \mathcal{N}\cap V \in \mathcal{N}^{V[G]}$ $\quad \mathcal{N}_V$

Let $\mathbb{A}_\varepsilon := \{T\subseteq 2^{<\omega} : T \text{ pruned tree}, \mu([T])>\varepsilon\}$. For every $0<\frac{1}{2}\leq\varepsilon<1$ $\mathbb{A}$ and $\mathbb{A}_\varepsilon$ are forcing equivalent (Truss in connections between amoeba algebras).

Let G_ε be an $\mathbb{A}_\varepsilon$-generic filter and $P_\varepsilon := \bigcap_{T\in G}[T]$. Then P_ε is a closed set with $\mu(P_\varepsilon)=\varepsilon$. Let N be a Lebesgue null set coded in V. ~~Then~~ and let $T\in\mathbb{A}_\varepsilon$. Then there is an open set $O\supseteq N$ s.t. $\mu([T]\cap O)>\varepsilon$. Let $S\subseteq 2^{<\omega}$ s.t. $[S]=[T]\cap O$. Then $S\leq T$ and $[S]\cap N=\emptyset$ and so $D_N:=\{T : [T]\cap N=\emptyset\}$ is dense.

$\Rightarrow P_\varepsilon\cap N=\emptyset$

$\Rightarrow \forall N\in\mathcal{N}_V\ \ N\cap P_\varepsilon=\emptyset$ $\qquad N\cap \overbrace{\bigcup_{n\in\omega} P_{1-\frac{1}{n}}}^{\mu=1}=\emptyset$

$\bigcup_{x\in\mathcal{N}_V} B_x = \omega^\omega\setminus\{x : x \text{ is } \mathbb{C}\text{-generic}\}$

Let $B\in\mathbb{P}_I$ $\quad \mathcal{N}$ is Borel generated $\Rightarrow B\setminus(\quad)\leq B$

2.) $\mathbb{D} * \mathbb{D}$ is an amoeba for $\mathbb{C}$ $\qquad$ $(\mathbb{C} * \mathbb{D}) \rightsquigarrow$ UM - generic

$UM := \{(\sigma, D) : \sigma \in (2^{<\omega})^{<\omega},\ D$ is open dense in $2^\omega\}$ $\qquad$ $= \{x \in 2^\omega : \sigma(i) \subseteq x\}$

$(\sigma, D) \leq (\tau, E) \iff \sigma \supseteq \tau,\ D \subseteq E,\ \underline{\forall i \in dom(\sigma) \setminus dom(\tau)\ \ O_{\sigma(i)} \subseteq E}$

$\mathcal{D} \setminus \mathcal{M}$

Claim UM is an amoeba for $\mathbb{C}$

Proof Let G be a UM-generic filter and let $x = \bigcup_{(\sigma, D) \in G} \sigma$. Then $x \in (2^{<\omega})^\omega$

We consider $\bigcap_{n \in \omega} \bigcup_{m \geq n} O_{x(m)} =: E$

Claim E is comeagre and for every $x \in \mathcal{M} \cap V$ $\ B_x \cap E = \emptyset$

Proof Claim 2: $\bigcup_{m \geq n} O_{x(m)}$ is open dense for every $n \in \omega$.

Proof Let $s \in 2^{<\omega}$ and $D_s^n := \{(\sigma, D) : \exists k \geq n\ \exists t \supseteq s\ \ \sigma(k) = t\}$

Let $(\sigma, D) \in UM$, since D is open dense there is $t \supseteq s$ s.t. $O_t \subseteq D$.

Extend σ to σ' s.t. $dom(\sigma') > n$ and $\exists k \geq n$ $\ \sigma'(k) = t$ and $\forall k \in dom(\sigma') \setminus dom(\sigma)$

$O_{\sigma'(k)} \subseteq D$. Then $(\sigma', D) \leq (\sigma, D)$ and $(\sigma', D) \in D_s^n$.

$\forall s \in 2^{<\omega}\ \exists t \supseteq s\ \ O_t \subseteq \bigcup_{m \geq n} O_{x(m)}$.

$\square$

$\Rightarrow$ E is comeagre.

Let N be closed null and coded in V. Then $D_N := \{(\sigma, D) : D \cap N = \emptyset\}$ is dense.
Hence, there is a $(\sigma, D) \in G$ s.t. $N \cap E = \emptyset$. Then for every $k \in \mathrm{dom}(x) \setminus \mathrm{dom}(\sigma)$
$O_{x}(k) \subseteq D$. Hence $\bigcup_{m > \mathrm{dom}(\sigma)} O_{x(m)} \cap N = \emptyset$ and so E is disjoint from N. $\square$

> $A \subseteq R$ is $\mathcal{E}$-rare iff $\forall T \in \mathbb{A}\ \exists S \leq T$ $\langle S \rangle \cap A = \emptyset$
>
> $\mathcal{E}$-meager iff it is a countable union of $\mathcal{E}$-rare sets.

$R := \{P \leq 2^{<\omega} : P \text{ is pruned tree} \wedge \mu([P]) = \frac{1}{2}\}$

Let $\pi : 2^{<\omega} \to \omega$ be the canonical bijection

$$\pi(P)(n) := \begin{cases} 1 & \pi^{-1}(n) \in P \\ 0 & \text{otherwise} \end{cases}$$

$\pi[R] \subseteq \omega^{\omega}$ We can show that $\pi[R]$ is G_δ $I_{\mathcal{E}}$

Let $T \in \mathbb{A}$ $\underbrace{\langle T \rangle}_{\text{region}} := \{P \in R : P \leq T\}$ $\mathcal{E} := \{\langle T \rangle : T \in \mathbb{A}\}$ is category base

Every region is closed. But not every closed is a region

Every region is non $\mathcal{E}$-~~rare~~ meagre

$A \subseteq \mathbb{R}$ is $\mathcal{E}$-regular iff $\forall T \in \mathbb{A}\ \exists S \subseteq T$ s.t. $\langle S \rangle \setminus A$ or $\langle S \rangle \cap A$ is $\mathcal{E}$-meagre

<u>Note</u> The $\mathcal{E}$-regular sets form a σ-algebra containing all Borel sets in $\mathbb{R}$.

A is $\mathcal{E}$-regular iff its $I_{\mathcal{E}}$-regular

~~$S = \{ s \in (2^{<\omega})^\omega ;$~~ $S = (2^{<\omega})^\omega$ c.c.c.

$x \in$ $p \perp q \Rightarrow \exists r \leq p\ \langle r \rangle \cap \langle q \rangle = \emptyset$

$\pi(x)(n) := \pi(x(n)) \in \omega^\omega$

$(\sigma, D) \in \mathbb{M}$ $\langle \sigma, D \rangle := \{ x \in S : \sigma \subseteq x$ and $\forall i \in \omega \setminus \mathrm{dom}(\sigma)\ O_{x(i)} \subseteq D \}$

$(\sigma, D) \leq (\tau, E)$ iff $\langle \sigma, D \rangle \subseteq \langle \tau, E \rangle$

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