1 Ikegami's Theorem for zero-dimensional Polish spaces

Let I be a σ-ideal on a set X. We call I proper if I contains all singletons but not the whole set. From now on every ideal shall be proper.

Let X be an uncountable Polish space and let I be a proper σ-ideal on X. We denote the partial order of all I-positive Borel sets in X ordered by inclusion by $\mathbb{P}_I$. Zapletal proved in [Zap08] that every forcing notion of this form adds a $\mathbb{P}_I$-generic element, i.e. an $x \in X$ such that there is a $\mathbb{P}_I$-generic filter G such that for every Borel set B coded in the ground model, $x \in B$ if and only if $B \in G$.

Let A be a subset of X. We say A is I-null if for every $B \in \mathbb{P}_I$ set, there is a $C \leq B$ such that $C \cap A = \emptyset$ and I-regular if for every $B \in \mathbb{P}_I$ set, there is a $C \leq B$ such that either $C \subseteq A$ or $C \cap A = \emptyset$. In [Kho12], Khomskii proved among other things a versions of Ikegami's Theorem for ideals living on the Baire space. We shall use his results to obtain a version of Ikegami's Theorem for σ-ideals living on zero-dimensional Polish spaces. In order to do so, for every proper σ-ideal I on a zero-dimensional Polish space we shall define a second σ-ideal I_* on the Baire space and use I_* to derive Ikegami's Theorem for I from Khomskii's results.

More precisely by [Kec95, Theorem 7.8], a zero-dimensional Polish space is homeomorphic to a closed subset of ω^ω. We therefore assume without loss of generality that all such spaces are subspaces of ω^ω. Let X be an uncountable, zero-dimensional Polish space and let I be a proper σ-ideal on X. We define

$$I_* := \{A \subseteq \omega^\omega : A \cap X \in I\}.$$

Lemma 1.1. *Let X be an uncountable, zero-dimensional Polish space and let I be a proper σ-ideal on X.*

1. *I_* is a proper σ-ideal on ω^ω.*
2. *If I is Borel generated, then I_* is also Borel generated.*
3. *$\mathbb{P}_I$ is a dense subset of $\mathbb{P}_{I_*}$ and so $\mathbb{P}_I$ and $\mathbb{P}_{I_*}$ are forcing equivalent.*
4. *A set of reals $A \subseteq \omega^\omega$ is I_*-null if and only if $A \cap X$ is I-null.*
5. *A set of reals $A \subseteq \omega^\omega$ is I_*-regular if and only if $A \cap X$ is I-regular.*

Proof. The first item follows directly, as I is a proper σ-ideal. We show the second item. Let $A \subseteq \omega^\omega$ be I_*-small. Then $A \cap X$ is an I-small. Since I is Borel generated, there is an I-small Borel set B which is a superset of $A \cap X$. Then $B \cup (\omega^\omega \setminus X)$ is an I_*-Borel set containing A.

The third item is clear, since for every I_*-positive Borel set B, $B \cap X$ is I-positive. The proof of the fourth and fifth items are similar. We only prove the fourth item and we start with the "if" direction. Let $A \subseteq \omega^\omega$ be a set of reals such that $A \cap X$ is I-null and let B be an I_*-positive Borel set. Then $B \cap X$ is an I-positive Borel set and so there is an I-positive Borel set $C \leq B \cap X$ which is disjoint from $A \cap X$. Furthermore, C is also an I_*-positive Borel set which is disjoint from A. We prove the "only if" direction. Let $A \subseteq \omega^\omega$ be an I_*-small set of reals and let B be an I-positive Borel set. Then B is also an I_*-positive Borel set and so there is an I_*-positive Borel set $C \leq B$ which is disjoint from A. Since C is a subset of X, C is also an I-positive Borel set. □

Before we can state Khomskii's version of Ikegami's Theorem we need a few additional definitions. We call an ideal *absolute* if for every inner model M of ZFC and every Borel set B coded in M, the statement $B \in I$ is absolute between V and M.

Let X be an uncountable, zero-dimensional Polish space, let I be a proper σ-ideal on X, and let M be an inner model of ZFC. An element of X is called *I-quasi-generic over M* if it omits all I-small Borel sets coded in M. The concept of quasi-generic was first introduced by Brendle, Halbeisen, and Löwe in [BHL05]. By definition, every $\mathbb{P}_I$-generic element over M is I-quasi-generic over M. The converse is true for forcing notion satisfying the c.c.c. The proof is the same as for I living in ω^ω (cf., [Kho12, Lemma 2.3.2]). Furthermore, since $\omega^\omega \setminus X$ is an I_*-small Borel set and I and I_* coincide on Borel sets in X, a real is I_*-quasi-generic over M if and only if it is I-quasi-generic over M.

A forcing notion $\mathbb{Q}$ is called *Σ^1_3-absolute*, if for every $\mathbb{Q}$-generic filter G, every Σ^1_3 formula is absolute between V and $V[G]$. Since $\mathbb{P}_I$ and $\mathbb{P}_{I_*}$ are forcing equivalent, $\mathbb{P}_{I_*}$ is Σ^1_3 absolute if and only if $\mathbb{P}_I$ is Σ^1_3-absolute.

Now, we can state Khomskii's version of Ikegami's Theorem. For a proof see [Kho12, Theorem 2.3.7 & Corollary 2.3.8].

Theorem 1.2 (Ikegami)**.** *Let I be a proper σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper and the set $\{c \in \mathsf{BC} : B_c \in I\}$ is $\boldsymbol{\Sigma}^1_2$. Then the following are equivalent:*

1. *Every $\boldsymbol{\Delta}^1_2$ set of reals is I-regular,*
2. *$\mathbb{P}_I$ is Σ^1_3-absolute, and*
3. *for every real $r \in \omega^\omega$ and every I-positive Borel set B, there is an I-quasi-generic real over $L[r]$.*

If $\mathbb{P}_I$ satisfies the c.c.c., then it is also equivalent to

4. *for every real $r \in \omega^\omega$, there is an $\mathbb{P}_I$-generic real over $L[r]$.*

Theorem 1.3 (Ikegami)**.** *Let I be a proper σ-ideal on ω^ω such that $\mathbb{P}_I$ is proper and the set $\{c \in \mathsf{BC} : B_c \in I\}$ is $\boldsymbol{\Sigma}^1_2$. Then the following are equivalent:*

1. *Every $\boldsymbol{\Sigma}^1_2$ set of reals is I-regular, and*
2. *for every real $r \in \omega^\omega$, the set $\{x \in \omega^\omega : x$ is not I-quasi-generic over $L[r]\}$ is I-null.*

If $\mathbb{P}_I$ satisfies the c.c.c. and I is Borel generated, then it is also equivalent to

3. *for every real $r \in \omega^\omega$, the set $\{x \in \omega^\omega : x$ is not $\mathbb{P}_I$-generic over $L[r]\}$ is I-small.*

We can use these theorems to proof similar characterization for our context:

Corollary 1.4. *Let X be an uncountable, zero-dimensional Polish space, let I be a proper σ-ideal on X such that $\mathbb{P}_I$ is proper and the set $\{c \in \mathsf{BC} : B_c \in I\}$ is $\boldsymbol{\Sigma}^1_2$. Then the following are equivalent:*

1. *Every $\boldsymbol{\Delta}^1_2$ subset of X is I-regular,*
2. *$\mathbb{P}_I$ is Σ^1_3-absolute, and*
3. *for every real $r \in \omega^\omega$ and every I-positive Borel set B, there is an I-quasi-generic real over $L[r]$.*

If $\mathbb{P}_I$ satisfies the c.c.c., then it is also equivalent to

4. *for every real $r \in \omega^\omega$, there is an $\mathbb{P}_I$-generic real over $L[r]$.*

Proof. By Lemma 1.1 and Theorem 1.2, we only have to check that $\{c \in \mathsf{BC} : B_c \in I_*\}$ is $\boldsymbol{\Sigma}^1_2$. But this follows directly from the fact that $\{c \in \mathsf{BC} : B_c \in I\}$ is $\boldsymbol{\Sigma}^1_2$. $\square$

Corollary 1.5. *Let X be an uncountable, zero-dimensional Polish space, let I be a proper σ-ideal on X such that $\mathbb{P}_I$ is proper and the set $\{c \in \mathsf{BC} : B_c \in I\}$ is $\boldsymbol{\Sigma}^1_2$. Then the following are equivalent:*

1. *Every $\boldsymbol{\Sigma}^1_2$ subset of X is I-regular, and*
2. *for every real $r \in \omega^\omega$, the set $\{x \in X : x$ is not I-quasi-generic over $L[r]\}$ is I-null.*

If $\mathbb{P}_I$ satisfies the c.c.c. and I is Borel generated, then it is also equivalent to

3. *for every real $r \in \omega^\omega$, the set $\{x \in \omega^\omega : x$ is not $\mathbb{P}_I$-generic over $L[r]\}$ is I-small.*

Proof. Follows directly from Lemma 1.1 and Theorem 1.3. $\square$

2 Characterization theorems for amoeba forcing

In this section, we use the generalized version of Ikegami's Theorem to prove characterization results for amoeba forcing. Amoeba forcing was first introduced by Martin and Solovay in [MS70] to prove that Martin's axiom implies that $\mathrm{add}(\mathcal{N}) = 2^\omega$, where $\mathcal{N}$ is the Lebesgue null ideal.

Amoeba Forcing is the partial order of all pruned trees T on 2 such that $\mu([T]) > \frac{1}{2}$, ordered by inclusion. We denote amoeba forcing by $\mathbb{A}$. Amoeba forcing satisfies the c.c.c. A prove can be found e.g., in [Kun11, pages 179f.].

In the following, we introduce a zero-dimensional Polish space $\mathbf{R}$ and define a regularity property on $\mathbf{R}$. Let $\mathbf{R}$ be the collection of all pruned trees P on 2 such that $\mu([P]) = \frac{1}{2}$ and let π be the canonical bijection between $2^{<\omega}$ and ω. We extend π to a function from pruned trees on 2 to ω^ω. Let T be a pruned tree on 2 and let $n \in \omega$. We define $\pi(T)$ as follows:

$$\pi(T)(n) := \begin{cases} 1 & \pi^{-1}(n) \in T, \\ 0 & \textit{otherwise.} \end{cases}$$

Then π codes pruned trees on 2 as real numbers. Let y be a code for a pruned tree. We denote the pruned tree coded by y by T_y.

Lemma 2.1. *Let y be a real.*

1. *The statement "y is a code for a pruned tree on 2" is Π^0_2.*
2. *If y is the code for a pruned tree on 2, then for every $p, q \in \omega$ with $q \neq 0$, the statements "$\mu([T_y]) \geq \frac{p}{p}$" and "$\mu([T_y]) \leq \frac{p}{q}$" are Π^0_2.*

Proof. We start with the first item. A real y is a code for a pruned tree on 2 if and only if

(a) $y \in 2^\omega$,

(b) T_y is nonempty,

(c) $\forall n \in \omega(y(n) = 1 \to \forall m < n(p^{-1}(m) \subseteq \pi^{-1}(n) \to y(m) = 1))$ (T_y is a tree on 2), and

(d) $\forall n \in \omega(y(n) = 1 \to \exists m \in \omega(\pi^{-1}(n) \subsetneq \pi^{-1}(m) \wedge y(m) = 1))$ (T_y is pruned).

Since all these statements are Π^0_2, the whole statement is also Π^0_2.

We prove the second item. let y be a code for a pruned tree on 2. Then $\mu([T_y]) \geq \frac{p}{q}$ if and only if for every finite sequence $s_0, \ldots, s_m \in 2^{<\omega} \setminus T_y$ with $|s_0| = \cdots = |s_m|$ and $s_i \neq s_j$ for $i \neq j$, $\frac{m+1}{2^{|s_0|}} \leq \frac{p}{q}$. Since the second statement is Π^0_1, the statement "$\mu([T_y]) \geq \frac{p}{q}$" is Π^0_1 and especially Π^0_2. Furthermore, $\mu([T_y]) \leq \frac{p}{q}$ if and only if for every $n > 0$, there is a finite sequence $s_0, \ldots, s_m \in 2^{<\omega} \setminus T_y$ with $|s_0| = \cdots = |s_m|$ and $s_i \neq s_j$ for $i \neq j$, $\frac{m+1}{2^{|s_0|}} \geq \frac{p}{q} - \frac{1}{n}$. Hence, the statement "$\mu([T_y]) \leq \frac{p}{q}$" is Π^0_2. □

By Lemma 2.1, the image $\pi[\mathbf{R}]$ is a G_δ subset of ω^ω. Hence, $\pi[\mathbf{R}]$ equipped with the subset topology is an uncountable, zero-dimensional Polish space. Then $\mathbf{R}$ with the topology induced by π is an uncountable, zero-dimensional Polish space.

Let $T \in \mathbb{A}$. We define $\langle T \rangle := \{P \in \mathbf{R} : P \subseteq T\}$ and $\mathcal{C} := \{\langle T \rangle : T \in \mathbb{A}\}$. The elements of $\mathcal{C}$ are called *regions*.

A set $A \subseteq \mathbf{R}$ is called *$\mathcal{C}$-rare* if for every $T \in \mathbb{A}$ there is an $S \leq T$ such that $\langle S \rangle \cap A = \emptyset$ and *$\mathcal{C}$-meager* if A is a countable union of $\mathcal{C}$-rare sets. The $\mathcal{C}$-meager sets form a proper σ-ideal on $\mathbf{R}$ (cf., [JR95, Section 1c]). We denote the σ-ideal by $I_\mathcal{C}$. Furthermore, $\mathbb{A}$ and $\mathcal{B}/I_\mathcal{C}$ are forcing equivalent (cf., [JR95, Corollary 1.6]).

Lemma 2.2.

1. *Every region is closed non-$\mathcal{C}$-meager.*
2. *$I_\mathcal{C}$ is Borel generated.*
3. *The set $\{c \in \mathsf{BC} : B_c \in I_\mathcal{C}\}$ is $\mathbf{\Sigma}^1_2$.*

Proof. We start with the first item. That every region is non-$\mathcal{C}$-meager was proved in [JR95, Lemma 1.3]. We check that every region is closed. Let $T \in \mathbb{A}$, let $P \notin \langle T \rangle$, and let x and y be codes for T and P, respectively. Then there is an $n \in \omega$ with $x(n) = 0$ and $y(n) = 1$. Let $s = y{\restriction}(n+1)$. Then $O_s \cap \mathbf{R}$ is open in $\mathbf{R}$, contains P, and is disjoint from $\langle T \rangle$. Hence, $\langle T \rangle$ is closed in $\mathbf{R}$.

We prove the second item. It is enough to show the $\mathcal{C}$-rare sets are Borel generated. Let $A \subseteq \mathbf{R}$ be $\mathcal{C}$-rare. Then the set $D := \{T \in \mathbb{A} : \langle T \rangle \cap A = \emptyset\}$ is dense in $\mathbb{A}$. Let $\mathcal{A} \subseteq D$ be a maximal antichain. Since $\mathbb{A}$ satisfies the c.c.c., $\mathcal{A}$ is countable. Hence, $\bigcup_{T \in \mathcal{A}} \langle T \rangle$ is a Borel set and disjoint from A. We show that $B := \mathbf{R} \setminus \bigcup_{T \in \mathcal{A}} \langle T \rangle$ is $\mathcal{C}$-rare. Let $T \in \mathbb{A}$. Then there is an element in $\mathcal{A}$ which is compatible with T. Let $S \in \mathbb{A}$ be a witness. Then $S \leq T$ and $\langle S \rangle$ is disjoint from A.

We prove the third item. Let $c \in 2^\omega$ be a Borel code with $B_c \subseteq \mathbf{R}$. We have already shown in the proof of the last item that a set A is $\mathcal{C}$-rare if and only if there is a maximal antichain $\mathcal{A}$ such that $\bigcup_{T \in \mathcal{A}} \langle T \rangle$ is disjoint from A. Hence, B_c is $\mathcal{C}$-meager if and only if there are antichains $\mathcal{A}_n$ such that B_c is a subset of $\bigcup_{n \in \omega} (\mathbf{R} \setminus \bigcup_{T \in \mathcal{A}_n} \langle T \rangle)$. Since $\mathbb{A}$ satisfies the c.c.c., B_c is $\mathcal{C}$-meager if and only if there are pruned trees T_{ij} on 2 such that:

1. $\mu([T_{ij}]) > \frac{1}{2}$ for every $i, j \in \omega$,
2. $\mathcal{A}_i := \{T_{ij} : j \in \omega\}$ is a maximal antichain for every $i \in \omega$, and
3. for every $x \in B_c$, there is an $i \in \omega$ such that $x \in \mathbf{R} \setminus \bigcup_{j \in \omega} \langle T_{ij} \rangle$.

By Lemma 2.1, the first statement is Π^0_2 and the second statement is Π^1_1. Furthermore, the third statement is also Π^1_1. Since we can code trees as reals, the whole statement is Σ^1_2. Therefore, $I_{\mathcal{C}}$ is $\mathbf{\Sigma}^1_2$. □

A set $A \subseteq \mathbf{R}$ is *$\mathcal{C}$-Baire* if for every $T \in \mathbb{A}$ there is an $S \leq T$ such that either $\langle S \rangle \setminus A$ or $\langle S \rangle \cap A$ is $\mathcal{C}$-meager. The $\mathcal{C}$-Baire sets form a σ-algebra on $\mathbf{R}$ containing all Borel sets.

Lemma 2.3. *A set is $\mathcal{C}$-Baire if and only if it is $I_{\mathcal{C}}$-regular.*

Proof. We start with a claim.

Claim 2.4. *A set $A \subseteq \mathbf{R}$ is $\mathcal{C}$-meager if and only if for every $T \in \mathbb{A}$ there is an $S \leq T$ such that $\langle S \rangle \cap A$ is $\mathcal{C}$-meager.*

Proof. The "only if" direction is clear. We prove the "if" direction. Let $A \subseteq \mathbf{R}$ such that for every $T \in \mathbb{A}$ there is an $S \leq T$ such that $\langle S \rangle \cap A$ is $\mathcal{C}$-meager. Then the set $D := \{T \in \mathbb{A} : \langle T \rangle \cap A \in I_{\mathcal{C}}\}$ is dense. Let $\mathcal{A} \subseteq D$ be a maximal antichain. Since $\mathbb{A}$ satisfies the c.c.c., $\mathcal{A}$ is countable. Hence, $M := \bigcup_{T \in \mathcal{A}} (\langle T \rangle \cap A)$ is $\mathcal{C}$-meager. We consider $A \setminus M$. Let $T \in \mathbb{A}$. Then there is an element in $\mathcal{A}$ which is compatible with T. Let S be a witness. Then $S \leq T$ and $S \cap A \subseteq M$. Therefore, $A \setminus M$ is $\mathcal{C}$-rare and so A is $\mathcal{C}$-meager. □

We prove the "if" direction. Let $A \subseteq \mathbf{R}$ be a set which is $I_{\mathcal{C}}$-regular and let $T \in \mathbb{A}$. Since $\langle T \rangle$ is a closed $I_{\mathcal{C}}$-positive set, there is an $I_{\mathcal{C}}$-positive Borel set $B \leq \langle T \rangle$ such that either $B \cap A = \emptyset$ or $B \subseteq A$. Furthermore, B is $\mathcal{C}$-Baire. By Claim 2.4, there is an $S \in \mathbb{A}$ such that $\langle S \rangle \setminus B$ is $\mathcal{C}$-meager. Then either $\langle S \rangle \cap A$ or $\langle S \rangle \setminus A$ is $\mathcal{C}$-meager. If T and S are compatible, then we are done. We suppose for a contradiction that T and S are incompatible. Then $\mu([T] \cap [S]) \leq \frac{1}{2}$. We remove a small Lebesgue measure positive subset of $[T]$ from $[S]$ to find an $S' \leq S$ such that $\mu([S'] \cap [T]) < \frac{1}{2}$. Then $\langle S' \rangle$ is disjoint from $\langle T \rangle$. Hence, $\langle S' \rangle \subseteq \langle S \rangle \setminus \langle T \rangle \subseteq \langle S \rangle \setminus B \in I_{\mathcal{C}}$. But this is a contradiction. Therefore, A is $\mathcal{C}$-Baire.

We prove the "only if" direction. Let $A \subseteq \mathbf{R}$ be $\mathcal{C}$-Baire and let B be a non-$\mathcal{C}$-meager Borel set. As before, there is a $T \in \mathbb{A}$ with $\langle T \rangle \setminus B$ is $\mathcal{C}$-meager. Then there is an $S \leq T$ such that either $\langle S \rangle \cap A$ or $\langle S \rangle \setminus A$ is $\mathcal{C}$-meager. Without loss of generality, we assume $\langle S \rangle \cap A$ is $\mathcal{C}$-meager. Since $I_{\mathcal{C}}$ is Borel generated, there is an $I_{\mathcal{C}}$-meager Borel set C containing $\langle S \rangle \cap A$. Then $\langle S \rangle \setminus C \leq B$ is a non-$\mathcal{C}$-meager Borel set and is disjoint from A.

In the case $\langle S \rangle \setminus A$ is $\mathcal{C}$-meager, we deduce that there is a $C \leq B$ which is a subset of A with the same argument. □

Let G be an $\mathbb{A}$-generic filter. Then $\bigcap G$ is a pruned tree on 2 and $\mu([\bigcap G]) = \frac{1}{2}$. Hence, $\bigcap G$ is an element of $\mathbf{R}$. We call such an element *amoeba real*.

Lemma 2.5. *Let M be an inner model of* ZFC*. Then an element of $\mathbf{R}$ is an amoeba real over M if and only if it is $I_{\mathcal{C}}$-quasi-generic over M.*

Proof. We start with the "if" direction. let $P \in \mathbf{R}$ be $I_{\mathcal{C}}$-quasi-generic over M. Then P is $\mathbb{P}_{I_{\mathcal{C}}}$-generic over M and so $G_P := \{B \in \mathbb{P}_{I_{\mathcal{C}}} : P \in B\}$ is a $\mathbb{P}_{I_{\mathcal{C}}}$-generic filter over M. One can easily check that

$$\begin{aligned} i : \mathbb{P}_{I_{\mathcal{C}}} &\longrightarrow (\mathcal{B}(\mathbf{R}/I_{\mathcal{C}}))^+, \\ B &\longmapsto \text{the equivalence class of } B \end{aligned}$$

is a dense embedding in M. Hence, $\tilde{i}(G_P) := \{A \in (\mathcal{B}(\mathbf{R}/I_\mathcal{C}))^+ : \exists B \in G_P\ i(B) \leq A\}$ is a $(\mathcal{B}(\mathbf{R}/I_\mathcal{C}))^+$-generic filter over M. By Lemma 2.2, Claim 2.4, and the fact that every Borel set is $\mathcal{C}$-Baire, the map

$$j : \mathbb{A} \longrightarrow (\mathcal{B}(\mathbf{R}/I_\mathcal{C}))^+,$$
$$T \longmapsto \text{the equivalence class of } \langle T\rangle$$

is also a dense embedding in M. Thus, $H_P := j^{-1}(\tilde{i}(G_P))$ is an $\mathbb{A}$-generic filter over M. It is enough to show that for every $T \in H_P$, P is a subset of T. We suppose for a contradiction that there is some $T \in H_P$ such that $P \nsubseteq T$. Then there is an $A \in \tilde{i}(G_P)$ such that $\langle T\rangle \in A$. Since $A \in \tilde{i}(G_P)$, there is a $B \in G_P$ such that $i(B) \leq A$. Hence, $B \setminus \langle T\rangle$ is $\mathcal{C}$-meager and so $P \notin B \setminus \langle T\rangle$. But this is a contradiction to $P \in B$. Therefore, $P \subseteq T$ for every $T \in H_P$ and so $P = \bigcap H_P$ is an amoeba real over M.

We prove the "only if" direction. Let P be an amoeba real over M. Then there is an $\mathbb{A}$-generic filter over M such that $P \bigcap G$. We suppose for a contradiction that there is a $\mathcal{C}$-meager Borel set $B \subseteq \mathbf{R}$ coded in M such that $P \in B$. Then there are $\mathcal{C}$-rare sets N_n such that $B = \bigcup_{n\in\omega} N_n$. Let $n \in \omega$ such that $P \in N_n$. We define $D_n := \{T \in \mathbb{A} : \langle T\rangle \cap N_n = \emptyset\}$. Then D_n is dense in M and so there is a $T \in G \cap D_n$. Then $P \in \langle T\rangle$ and $\langle T\rangle$ is disjoint from B_n but this is a contradiction to $P \in B_n$. Therefore, P is $I_\mathcal{C}$-quasi-generic over M. □

Now, we can prove characterization results for amoeba forcing:

Theorem 2.6. *The following are equivalent:*

1. *Every $\mathbf{\Delta}^1_2$ subset of $\mathbf{R}$ is $\mathcal{C}$-Baire,*
2. *$\mathbb{A}$ is Σ^1_3-absolute, and*
3. *for every real $r \in \omega^\omega$, there is an amoeba real over $L[r]$.*

Proof. Follows from Corollary 1.4, Lemma 2.2, Lemma 2.3, and Lemma 2.5. □

Theorem 2.7. *The following are equivalent:*

1. *Every $\mathbf{\Sigma}^1_2$ subset of $\mathbf{R}$ is $\mathcal{C}$-Baire, and*
2. *for every real $r \in \omega^\omega$, the set $\{P \in \mathbf{R} : P$ is not an amoeba real over $L[r]\}$ is $\mathcal{C}$-meager.*

Proof. Follows from Corollary 1.5, Lemma 2.2, Lemma 2.3, and Lemma 2.5. □

3 Amoeba forcing and Lebesgue measurability

There are several results about the connection between amoeba forcing and Lebesgue measurability; e.g., Judah proved in [Jud93] the following fact:

Fact 3.1 (Judah)**.** *Every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable if and only if $\mathbb{A}$ is Σ^1_3-absolute.*

Fact 3.1 together with Theorem 2.6 gives us the following Theorem:

Theorem 3.2. *The following are equivalent:*

1. *Every* $\mathbf{\Sigma}^1_2$ *set of reals is Lebesgue measurable,*
2. *for ever real* $r \in \omega^\omega$*, the set* $\{x \in \omega^\omega : x$ *is not random over* $L[r]\}$ *is Lebesgue null,*
3. *every* $\mathbf{\Delta}^1_2$ *subset of* $\mathbf{R}$ *is* $\mathcal{C}$*-Baire,*
4. $\mathbb{A}$ *is* Σ^1_3*-absolute, and*
5. *for every real* $r \in \omega^\omega$*, there is an amoeba real over* $L[r]$*.*

This result is not new, but it was never properly documented. In the following, we give an alternative proof of the direction 1. $\Rightarrow$ 5 using covering reals. The idea of this proof is similar to [BJ92, Theorem 4.1.2].

Let $\mathcal{S}_0 := \{f : f(n) \subseteq 2^n \text{ and } \sum_{n\in\omega} |f(n)|2^{-n} < \frac{1}{2}\}$. We define a partial order on $\mathcal{S}_0$ as follows:

$$g \leq^* f :\Leftrightarrow \exists m \in \omega\ \forall n > m\ g(n) \subseteq f(n).$$

An $f \in \mathcal{S}_0$ is called *covering real* if $g \leq^* f$ for every $f \in \mathcal{S}_0$. We use covering reals to prove the existence of amoeba reals. In order to do so, we need a fact that was proven by Truss, cf., [Tru77, Lemma 6.3].

Fact 3.3 (Truss)**.** *Let* M *be a transitive model of* ZFC*. Suppose there is an open set* $O \subseteq \omega^\omega$ *with* $\mu(O) < \frac{1}{2}$ *such that for every open set* $U \subseteq \omega^\omega$ *coded in* M *with* $\mu(U) < \frac{1}{2}$*, there is a finite union of basic open sets* U' *satisfying* $O \cup U = O \cup U'$*. Then for any Cohen real* x *over* $M[O]$*, there is an amoeba real over* M *in* $M[O][x]$*.*

We shall use this fact to prove a similar statement for covering reals.

Lemma 3.4. *Let* M *be a transitive model of* ZFC*. If* f *is a covering real over* M*, then for any Cohen real* x *over* $M[f]$*, there is an amoeba real over* M *in* $M[f][x]$*.*

Proof. Let f be a covering real over M and let $O := \bigcup_{n\in\omega} \bigcup_{s\in f(n)} O_s$. Then O is a open set with $\mu(O) < \frac{1}{2}$. We shall show that O satisfies the requirements for Lemma 3.3. Let U be an open set coded in M with $\mu(U) < \frac{1}{2}$. We define recursively:

$$\begin{aligned} g(0) &:= \emptyset \\ g(n+1) &:= \{s \in 2^{n+1} : \forall k \leq n\ s{\restriction}k \notin g(k) \text{ and } O_s \subseteq B\}. \end{aligned}$$

Then $g \in \mathcal{S}_0 \cap M$ and $\bigcup_{n\in\omega} \bigcup_{s\in g(n)} O_s = U$. Since f is a covering real over M, $g \leq^* f$. Hence, there is an $m \in \omega$ such that for every $n > m$, $g(n)$ is a subset of $f(n)$. Let $U' := \bigcup_{n\leq m} \bigcup_{s\in g(n)} O_s$. Then U' is a finite union of basic open sets and $U' \subseteq U$. Hence, $O \cup U'$ is a subset of $O \cup U$. Let $x \in U$. Then there is an $n \in \omega$ and an $s \in g(n)$ such that $x \in O_s$. We make a case distinction:

Case 1: $n \leq m$. Then x is an element of U'.

Case 2: $n > m$. Then $g(n)$ is a subset of $f(n)$. Hence, x is an element of O.

In both cases, x is an element of $O \cup U'$. Therefore, $O \cup U = O \cup U'$ and we are done by Fact 3.3. □

Before we can prove that $\mathbf{\Sigma}^1_2(\mathcal{N})$ implies for every real $r \in \omega^\omega$, there is an amoeba real over $L[r]$ we need an additional lemma.

Lemma 3.5 (Bartoszyński, Beese). *There are functions $\alpha_0 : \mathcal{S}_0 \to \mathcal{N}$ and $\alpha_0^* : \mathcal{N} \to \mathcal{S}_0$ such that $\alpha_0(f) \subseteq N$ implies $f \leq^* \alpha_0^*(N)$ for all $f \in \mathcal{S}_0$ and $N \in \mathcal{N}$. Additional, if f is in $L[r]$, then $\alpha_0(f)$ is a Borel set code in $L[r]$.*

Proof. Let $\{O^i_j \subseteq 2^\omega : i, j \in \omega\}$ be a collection of open sets in 2^ω such that $\mu(O^i_j) = 2^{-i}$ and $\mu(O^i_j \cap O^k_l) = \mu(O^i_j) \cdot \mu(O^k_l)$ for every $i, j, k, l \in \omega$. This is possible cf., e.g., [BJ95, Lemma 1.3.23]. Furthermore, for every $i \in \omega$ let $\{s^i_j : j < 2^i\}$ be an enumeration of 2^i.

First, we define $\alpha_0(f)$. Let $f \in \mathcal{S}_0$. We define

$$\alpha_0(f) := \bigcap_{k\in\omega} \bigcup_{i \geq k} \bigcup_{s^i_j \in f(i)} O^i_j.$$

Then $\alpha_0(f)$ is a Borel Lebesgue null set and if f is coded in $L[r]$, then $\alpha_0(f)$ is also coded in $L[r]$.

Next, we define α_0^*. Let N be a Lebesgue null set and let $K^N \subseteq 2^\omega$ be a compact set with positive Lebesgue measure such that K^N and N are disjoint. Without loss of generality we can assume that for every open set $U \subseteq 2^\omega$ that meets K^N, $\mu(K^N \cap U)$ is positive. Let $\{U_n : n \in \omega\}$ be an enumeration of all basic open sets that meets K^N. We define $F^N_n(i) := \{s^i_j : K^N \cap U_n \cap O^i_j = \emptyset\} \subseteq 2^i$ for $i, n \in \omega$.

For every $n \in \omega$, it holds that

$$0 < \mu(K^N \cap U_n) \leq \mu\left(\bigcap_{i\in\omega} \bigcap_{s^i_j \in F^N_n(i)} (2^\omega \setminus O^i_j)\right) = \prod_{i\in\omega} (1 - 2^{-i})^{\left|F^N_n(i)\right|}.$$

By a standard convergence test, this is equivalent to $\sum_{i\in\omega} |F^N_n(i)| 2^{-i}$ converges. Then there is a $k^N_n \in \omega$ such that $\sum_{i\in\omega\setminus k^N_n} |F^N_n(i)| 2^{-i} < \frac{1}{2}$. We define for $i \in \omega$

$$G^N_n(i) := \begin{cases} \emptyset & i < k^N_n \\ F^N_n(i) & i \geq k^N_n. \end{cases}$$

Then $G^N_n \in \mathcal{S}_0$ for $n \in \omega$. We construct a $g^N \in \mathcal{S}_0$ such that $G^N_n \leq^* g^N$ for every $n \in \omega$. We start with $g^N_0 := G^N_0$. Let $0 < \varepsilon < \frac{1}{2}$ such that $\sum_{i\in\omega} |g^N_0(i)| 2^{-i} < \varepsilon$. We assume g^N_n is already constructed with $\sum_{i\in\omega} |g^N_n(i)| 2^{-i} < \varepsilon$. Then there is an $\varepsilon_n > 0$ such that $\sum_{i\in\omega} |g^N_n(i)| 2^{-i} + \varepsilon_n < \varepsilon$. Since $\sum_{i\in\omega} |G^N_{n+1}(i)| 2^{-i}$ converges, there is an $i_{n+1} \in \omega$ such that $\sum_{i\in\omega\setminus i_{n+1}} |G^N_{n+1}(i)| 2^{-i} < \varepsilon_n$. We define for $i \in \omega$

$$g^N_{n+1}(i) := \begin{cases} g^N_n(i) & i < i_{n+1} \\ g^N_n(i) \cup G^N_{n+1}(i) & i \geq i_{n+1}. \end{cases}$$

Then $\sum_{i\in\omega} |g^N_n(i)| 2^{-i} < \varepsilon$ for every $n \in \omega$. We define g^N by $g^N(i) := g^N_n(i)$ for some $n \in \omega$ with $i < i_n$. Then $\sum_{i\in\omega} |g^N(i)| 2^{-i} \leq \varepsilon < \frac{1}{2}$ and so $g^N \in \mathcal{S}_0$. Furthermore, $G^N_n \leq^* g^N$ for every $n \in \omega$. We set $\alpha_0^*(N) := g^N$.

We check that α_0 and α_0^* satisfy the desired properties. Let $f \in \mathcal{S}_0$ and $N \subseteq 2^\omega$ be Lebesgue null such that $\alpha_0(f) \subseteq N$. Then $\alpha_0(f)$ is disjoint from K^N and so

$$K^N \cap \bigcap_{k\in\omega} \bigcup_{i\geq k} \bigcup_{s^i_j \in f(i)} O^i_j = \emptyset.$$

Since K^N is compact and $\alpha_0(f)$ is G_δ, by Baire's category theorem there is an $m \in \omega$ such that $K^N \cap \bigcup_{i \geq m} \bigcup_{s^i_j \in f(i)} O^i_j$ is not dense in K^N. Hence, there is $n \in \omega$ such that $U_n \cap K^N$ is disjoint from $K^N \cap \bigcup_{i \geq m} \bigcup_{s^i_j \in f(i)} O^i_j$. Then $f(i) \subseteq G^N_n(i)$ for every $i > \max\{m, k^N_n\}$. Therefore, $f \leq^* G^N_n \leq^* \alpha^*_0(N)$. □

Theorem 3.6. *If every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable, then for every real $r \in \omega^\omega$, there is an amoeba real over $L[r]$.*

Proof. Let $r \in \omega^\omega$ be a real. Since every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable, the set $N_r := \{x \in \omega^\omega : x \text{ is not random over } L[r]\}$ is Lebesgue null. Let $g \in \mathcal{S}_0 \cap L[r]$. By Lemma 3.5, $\alpha_0(g)$ is a Borel set coded in $L[r]$ and so $\alpha_0(g)$ is a subset of N_r. Again by Lemma 3.5, $g \leq^* \alpha^*_0(N_r) =: f$. Hence, $g \leq^* f$ for every $g \in \mathcal{S}_0 \cap L[r]$. Therefore, f is a covering real over $L[r]$. Since every $\mathbf{\Sigma}^1_2$ set of reals is Lebesgue measurable, for every $a \in \omega^\omega$, there is a Cohen real over $L[a]$. Hence, there is a Cohen real over $L[r, f]$. By Lemma 3.4, there is an amoeba real over $L[r]$. □

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