Generalising the Closed Null Ideal

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We will be working with the Cantor space ${}^{\omega}2$, with the product topology and 2 having the discrete topology. Partial functions $s\colon \omega \rightharpoonup 2$ with finite $\mathrm{dom}(s)$ provide basic clopen sets

$$[s] = \{f \in {}^{\omega}2 \mid s \subseteq f\}.$$

Give $2 = \{0,1\}$ the coin-flip measure: $\mu(\{0\}) = \mu(\{1\}) = \frac{1}{2}$, then this extends to a product measure on basic clopen $[s]$:

$$\mu([s]) = \frac{1}{2^{|\mathrm{dom}(s)|}}.$$

This extends to a *Lebesgue measure* on ${}^{\omega}2$. A subset $N \subseteq {}^{\omega}2$ is called *(Lebesgue) null* if there exists a measurable superset $N' \supseteq N$ with $\mu(N') = 0$.

Let $\mathcal{N}$ denote the family of null sets of ${}^{\omega}2$, then $\mathcal{N}$ is a σ-*ideal*, i.e., closed under subsets and countable unions.

Let $\mathcal{E}$ be the σ-ideal consisting of countable unions of *closed null sets*, i.e., of sets that are both null and closed.

Let $\mathcal{NWD}$ be the family of *nowhere dense* subsets of ${}^{\omega}2$; a set $N \subseteq {}^{\omega}2$ is *nowhere dense* if for any basic clopen $[s]$ there is basic clopen $[t] \subseteq [s]$ with $N \cap [t] = \varnothing$. Let $\mathcal{M}$ be the σ-ideal of *meagre* sets, i.e., of countable unions of nowhere dense sets.

Proposition

$\mathcal{E} \subseteq \mathcal{N} \cap \mathcal{M}$.

Proof. Suppose E is closed and dense in $[s]$, then $[s] \subseteq E$ because E is closed, hence $\mu(E) \geq \mu([s]) > 0$. $\square$

Let IP denote the set of interval partitions of ω. Given $I = \langle I_n \mid n \in \omega \rangle \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}\left({}^{I_n}2\right)$, we write $[\varphi]_* = \{x \in {}^\omega 2 \mid \forall^\infty n \in \omega (x \restriction I_n \in \varphi(n))\}$.

Theorem *Bartoszyński and Shelah 1992, Theorem 4.3*

The following are equivalent for $X \subseteq {}^\omega 2$.

1. $X \in \mathcal{E}$,
2. There is $I = \langle I_n \mid n \in \omega \rangle \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}\left({}^{I_n}2\right)$ such that $X \subseteq [\varphi]_*$ and $\sum_{n\in\omega} \frac{|\varphi(n)|}{2^{|I_n|}}$ converges.

Remark

If $B = \langle B_n \mid n \in \omega \rangle$ is a partition of ω into finite sets, then we may replace I by B in the above characterisation.

Let κ be a regular uncountable cardinal and $\kappa = \kappa^{<\kappa}$.

The *higher context* is the result of replacing ω by κ:

Classical	ω	${}^{\omega}\omega$	${}^{\omega}2$	$\mathcal{P}(\omega)$	finite	countable	$\aleph_1$
Higher	κ	${}^{\kappa}\kappa$	${}^{\kappa}2$	$\mathcal{P}(\kappa)$	$<\kappa$	$\leq\kappa$	κ^+

We will work mainly with ${}^{\kappa}2$, with the $<\kappa$*-box topology*, i.e., partial functions $s : \kappa \rightharpoonup 2$ with $|\mathrm{dom}(s)| < \kappa$ provide basic clopen sets

$$[s] = \{f \in {}^{\kappa}2 \mid s \subseteq f\}.$$

Instead of σ-ideals on ${}^{\omega}2$, we will work with $\leq\kappa$-complete ideals on ${}^{\kappa}2$, where $\mathcal{I}$ is $\leq\kappa$-complete if $\mathcal{I}$ is closed under unions of size κ.

Let $\mathcal{NWD}_\kappa$ be the family of nowhere dense subsets of ${}^\kappa 2$. Since κ is regular, $\mathcal{NWD}_\kappa$ is $<\kappa$-complete, but note that it is not $\leq\kappa$-complete. For example

$$\{f \in {}^\kappa 2 \mid \forall^\infty \alpha \in \kappa(f(\alpha) = 0)\}$$

is a dense set of cardinality $\kappa = 2^{<\kappa}$, and thus the union of κ many singletons.

Let $\mathcal{M}_\kappa$ be the least $\leq\kappa$-complete ideal extending $\mathcal{NWD}_\kappa$, then the elements of $\mathcal{M}_\kappa$ will be called κ-*meagre* sets.

The reals $\mathbb{R}$ form a particularly nice field that allows for a particularly nice theory of infinite summation. This is used to define (Lebesgue) measure, because we can make sense of the property of σ-additivity.

This does *not* generalise nicely to higher Baire spaces, as there is no suitable generalisation of $\mathbb{R}$. This means we cannot define $\mathcal{N}$ by replacing ω with κ.

Closed sets $X \subseteq {}^{\omega}2$ are sets of branches $[T]$ of trees $T \subseteq {}^{<\omega}2$. For such T, note that $\mu([T]) = 0$ if and only if for each $s \in T$ there is $n_s \in \omega$ with $\mathrm{dom}(s) < n_s$ such that for at least *half* of the extensions $t \supseteq s$ with $\mathrm{dom}(t) = n_s$ we have $t \notin T$.

Define $T \subseteq {}^{<\kappa}2$ to be *halving* if every $s \in T$ has $\alpha_s \in \kappa$ with $\mathrm{dom}(s) < \alpha_s$ such that

$$|\{t \in {}^{\alpha_s}2 \mid s \subseteq t \in T\}| \leq |\{t \in {}^{\alpha_s}2 \mid s \subseteq t \notin T\}|.$$

Let us define $\mathcal{H}_\kappa$ to be the least ${\leq}\kappa$-complete ideal containing $[T]$ for every halving $T \subseteq {}^{<\kappa}2$.

Proposition

$\mathcal{H}_\kappa = \mathcal{M}_\kappa$.

Proof. Easily $[T] \in \mathcal{NWD}_\kappa$ for every halving $T \subseteq {}^{<\kappa}2$. We show that reversely $\mathcal{NWD}_\kappa \subseteq \mathcal{H}_\kappa$.

Let $N \in \mathcal{NWD}_\kappa$, and without loss we may assume N is closed. Let $T \subseteq {}^{<\kappa}2$ be such that $N = [T]$ and $s \in T$, then there is $t \in {}^{<\kappa}2$ with $s \subseteq t$ and $t \notin T$. We may assume without loss of generality that $\mathrm{dom}(t) \geq \mathrm{dom}(s) + \gamma$ for some infinite γ, then $\alpha_s = \mathrm{dom}(s) + \gamma \cdot 2$ suffices: there are $|{}^\gamma 2| = 2^{|\gamma|}$ extensions of t at height α_s, but also $\left|{}^{\gamma \cdot 2}2\right| = 2^{|\gamma|}$ extensions of s. $\square$

Hence, in some sense $\mathcal{M}_\kappa$ can be viewed as the higher closed null ideal.

Recall that $X \in \mathcal{E}$ iff $X \subseteq [\varphi]_*$ for some $\langle I_n \mid n \in \omega \rangle \in \mathrm{IP}$ and $\varphi \in \prod_{n\in\omega} \mathcal{P}({}^{I_n}2)$ such that $\sum_{n\in\omega} \frac{|\varphi(n)|}{2^{|I_n|}}$ converges.

Let BP_κ be the set of partitions $\langle B_\alpha \mid \alpha \in \kappa \rangle$ of κ into sets of size $<\kappa$ such that $\liminf_{\alpha\in\kappa} |B_\alpha| = \kappa$. Let $\mathrm{IP}_\kappa \subseteq \mathrm{BP}_\kappa$ consist of those $\langle I_\alpha \mid \alpha \in \kappa \rangle \in \mathrm{BP}_\kappa$ such that I_α is an interval for each α.

If $B = \langle B_\alpha \mid \alpha \in \kappa \rangle \in \mathrm{BP}_\kappa$, then we define:

$$\Sigma_\kappa(B) = \left\{ \varphi \in \prod_{\alpha\in\kappa} \mathcal{P}({}^{B_\alpha}2) \;\middle|\; |\varphi(\alpha)| < 2^{|B_\alpha|} \right\},$$

$$\mathcal{E}_\kappa(B) = \{ X \subseteq {}^\kappa 2 \mid \exists \varphi \in \Sigma_\kappa(B)(X \subseteq [\varphi]_*) \},$$

$$\mathcal{E}_\kappa^- = \bigcup_{I\in\mathrm{IP}_\kappa} \mathcal{E}_\kappa(I), \qquad \mathcal{BE}_\kappa^- = \bigcup_{B\in\mathrm{BP}_\kappa} \mathcal{E}_\kappa(B).$$

We let $\mathcal{E}_\kappa$ and $\mathcal{BE}_\kappa$ be the least $\leq\kappa$-complete ideals such that $\mathcal{E}_\kappa^- \subseteq \mathcal{E}_\kappa$ and $\mathcal{BE}_\kappa^- \subseteq \mathcal{BE}_\kappa$.

Theorem *Marton, Šupina, Repický, and vdV. 2025+*

$\mathcal{E}^-_\kappa \subsetneq \mathcal{BE}^-_\kappa$ and $\mathcal{E}_\kappa \subsetneq \mathcal{BE}_\kappa$.

Proof. Clearly $\mathrm{IP}_\kappa \subseteq \mathrm{BP}_\kappa$ implies the inclusions $\subseteq$. Define

$$X = \{x \in {}^\kappa 2 \mid \forall \alpha \in \kappa(x(2 \cdot \alpha + 1) = 0)\},$$

then we claim that $X \in \mathcal{BE}_\kappa \setminus \mathcal{E}_\kappa$. Let $B \in \mathrm{BP}_\kappa$ be such that each B_α contains exactly one *even* ordinal $\xi_\alpha = 2 \cdot \xi'_\alpha$, and let $\varphi(\alpha) = \left\{s \in {}^{B_\alpha}2 \;\middle|\; \forall \xi \in B_\alpha \setminus \{\xi_\alpha\}(s(\xi) = 0)\right\}$, then $X \subseteq [\varphi]_*$.

Let $I^\zeta \in \mathrm{IP}_\kappa$ and $\varphi^\zeta \in \Sigma_\kappa(I^\zeta)$ for each $\zeta \in \kappa$. For each $\alpha \in \kappa$ choose β_α, ζ_α such that $I_\alpha := I^{\zeta_\alpha}_{\beta_\alpha}$ is infinite, $I_\alpha \cap I_{\alpha'} = \varnothing$ when $\alpha \neq \alpha'$, and $|\{\alpha \in \kappa \mid \zeta = \zeta_\alpha\}| = \kappa$. Since I_α is an infinite interval, $|X \restriction I_\alpha| = 2^{|I_\alpha|}$, thus let $s_\alpha \in (X \restriction I_\alpha) \setminus \varphi^{\zeta_\alpha}(\beta_\alpha)$. So, if $x \in X$ with $\bigcup_{\alpha \in \kappa} s_\alpha \subseteq x$, then $x \notin [\varphi^\zeta]_*$ for any $\zeta \in \kappa$. $\square$

For $x \in {}^\kappa 2$ and $B = \langle B_\alpha \mid \alpha \in \kappa \rangle \in \mathrm{BP}_\kappa$, let us write

$$\begin{aligned} \langle x, B \rangle &= \{ y \in {}^\kappa 2 \mid \quad \forall \alpha \in \kappa \quad (y \restriction B_\alpha \neq x \restriction B_\alpha) \} \\ \langle x, B \rangle_* &= \{ y \in {}^\kappa 2 \mid \forall^\infty \alpha \in \kappa \; (y \restriction B_\alpha \neq x \restriction B_\alpha) \} \end{aligned}$$

Define $\mathcal{CM}_\kappa = \{ X \subseteq {}^\kappa 2 \mid \exists x \in {}^\kappa 2 \exists I \in \mathrm{IP}_\kappa (X \subseteq \langle x, I \rangle_*) \}$.

Theorem *Blass, Hyttinen, and Zhang, § 4*

$\mathcal{CM}_\kappa$ is a $\leq\kappa$-complete ideal, $\mathcal{CM}_\kappa \subseteq \mathcal{M}_\kappa$, and equality holds if and only if κ is a regular strong limit cardinal.

Proposition

If $B \in \mathrm{BP}_\kappa$, then there is $I \in \mathrm{IP}_\kappa$ and a sequence $\langle A_\alpha \mid \alpha \in \kappa \rangle$ of subsets of κ such that $I_\alpha = \bigcup_{\xi \in A_\alpha} B_\xi$.

Corollary

$\mathcal{CM}_\kappa = \{ X \subseteq {}^\kappa 2 \mid \exists x \in {}^\kappa 2 \exists B \in \mathrm{BP}_\kappa (X \subseteq \langle x, B \rangle_*) \}$.

Theorem *Marton, Šupina, Repický, and vdV. 2025+*

$\mathcal{BE}_\kappa \subseteq \mathcal{CM}_\kappa$.

Proof. Let $Y \in \mathcal{BE}_\kappa^-$, and let $B \in \mathrm{BP}_\kappa$ and $\varphi \in \Sigma_\kappa(B)$ be such that $Y \subseteq [\varphi]_*$. For each $\alpha \in \kappa$, let $x_\alpha \in {}^{B_\alpha}2 \setminus \varphi(\alpha)$ and define $x = \bigcup_{\alpha\in\kappa} x_\alpha$. If $y \in Y$, then $y \restriction B_\alpha \in \varphi(\alpha)$ for almost all α, hence $y \restriction B_\alpha \neq x \restriction B_\alpha$ for almost all α, thus $y \in \langle x, B\rangle_*$. Therefore $Y \in \mathcal{CM}_\kappa$. □

Overview

$$\begin{array}{ccccccc} \mathcal{E}_\kappa & \subsetneq & \mathcal{BE}_\kappa & \subseteq & \mathcal{CM}_\kappa & \subseteq & \mathcal{M}_\kappa \\ \cup\!\mid & & \cup\!\mid & & & & \\ \mathcal{E}_\kappa^- & \subsetneq & \mathcal{BE}_\kappa^- & & & & \end{array}$$

Let $I \in \mathrm{IP}_\kappa$ be such that $|I_\alpha| \geq 2$ for all $\alpha \in \kappa$ and let $x \in {}^\kappa 2$. If $S_\alpha = {}^{I_\alpha}2 \setminus \{x \restriction I_\alpha\}$, then $\langle x, I\rangle \subseteq {}^\kappa 2$ with the subspace topology and $\prod_{\alpha\in\kappa} S_\alpha$ with the product topology are homeomorphic. Define $\mathcal{M}_\kappa^{\langle x,I\rangle}$ as the ideal of κ-meagre subsets of $\langle x, I\rangle$, then $\mathcal{M}_\kappa^{\langle x,I\rangle}$ is proper, i.e., $\langle x, I\rangle$ is not the union of κ many nowhere dense subsets of $\langle x, I\rangle$.

Theorem *Marton, Šupina, Repický, and vdV. 2025+*

$\mathcal{BE}_\kappa \subsetneq \mathcal{CM}_\kappa$.

Proof. Let $x \in {}^\kappa 2$ and $I \in \mathrm{IP}_\kappa$ with $|I_\alpha| \geq 2$ for all $\alpha \in \kappa$, then $\langle x, I\rangle \in \mathcal{CM}_\kappa \setminus \mathcal{M}_\kappa^{\langle x,I\rangle}$. To conclude the proof we claim that if $A \in \mathcal{BE}_\kappa^-$, then $A \cap \langle x, I\rangle \in \mathcal{M}_\kappa^{\langle x,I\rangle}$, and thus $A' \cap \langle x, I\rangle \in \mathcal{M}_\kappa^{\langle x,I\rangle}$ for all $A' \in \mathcal{BE}_\kappa$, so $\langle x, I\rangle \notin \mathcal{BE}_\kappa$. $\cdots$

$\cdots$ Let $B \in \mathrm{BP}_\kappa$ and $\varphi \in \Sigma_\kappa(B)$ be such that $A \subseteq [\varphi]_*$. Define $[\varphi]_\beta = \{y \in {}^\kappa 2 \mid \forall \alpha \geq \beta(y \restriction B_\alpha \in \varphi(\alpha))\}$, then $[\varphi]_* = \bigcup_{\beta \in \kappa} [\varphi]_\beta$.

Claim. $[\varphi]_\beta$ is nowhere dense in $\langle x, I \rangle$.

Fix some $s \in {}^{<\kappa}2$ such that $\mathrm{dom}(s) = \bigcup_{\alpha \in \delta} I_\alpha$ for some $\delta \in \kappa$ and $[s] \cap \langle x, I \rangle \neq \varnothing$. Let $\gamma \geq \beta$ and $\delta' > \delta$ be such that B_γ is infinite and $B_\gamma \subseteq \bigcup_{\alpha \in [\delta, \delta')} I_\alpha$.

Let $S = \{\alpha \in [\delta, \delta') \mid I_\alpha \subseteq B_\gamma\}$, and for each $\alpha \in [\delta, \delta')$ pick $\xi_\alpha \in I_\alpha$ arbitrary, with $\xi_\alpha \notin B_\gamma$ if $\alpha \notin S$. Now note that $|B_\gamma \setminus \{\xi_\alpha \mid \alpha \in [\delta, \delta')\}| = |B_\gamma|$, so there is $t \in {}^{B_\gamma}2 \setminus \varphi(\gamma)$ such that $t(\xi_\alpha) \neq x(\xi_\alpha)$ for all $\alpha \in [\delta, \delta')$. We may now extend $s \cup t$ to some s' with $\mathrm{dom}(s') = \bigcup_{\alpha \in \delta'} I_\alpha$ where $s'(\xi_\alpha) \neq x(\xi_\alpha)$ for all $\alpha \in [\delta, \delta')$. Then $[s'] \cap \langle x, I \rangle \neq \varnothing$ and $[s'] \cap [\varphi]_\beta = \varnothing$. $\square$

We may conclude:

Theorem

$\mathcal{E}_\kappa \subsetneq \mathcal{BE}_\kappa \subsetneq \mathcal{CM}_\kappa = \mathcal{M}_\kappa$ if $\kappa = \kappa^{<\kappa}$ is strongly inaccessible.
$\mathcal{E}_\kappa \subsetneq \mathcal{BE}_\kappa \subsetneq \mathcal{CM}_\kappa \subsetneq \mathcal{M}_\kappa$ if $\kappa = \kappa^{<\kappa}$ is weakly inaccessible.

Since we require $\liminf_{\alpha\in\kappa} |B_\alpha| = \kappa$ in the definition of BP_κ, and thus of $\mathcal{BE}_\kappa$, our definition trivialises when $\kappa = \lambda^+ = 2^\lambda$ for some λ.

For a σ-ideal $\mathcal{I} \subseteq \mathcal{P}({}^{\omega}2)$, we define

$$\mathrm{cov}(\mathcal{I}) = \min\{|C| \mid C \subseteq \mathcal{I} \text{ and } \bigcup C = {}^{\omega}2\},$$

$$\mathrm{non}(\mathcal{I}) = \min\{|N| \mid N \subseteq {}^{\omega}2 \text{ and } N \notin \mathcal{I}\},$$

$$\mathrm{add}(\mathcal{I}) = \min\{|A| \mid A \subseteq \mathcal{I} \text{ and } \bigcup A \notin \mathcal{I}\},$$

$$\mathrm{cof}(\mathcal{I}) = \min\{|F| \mid F \subseteq \mathcal{I} \text{ and } \forall I \in \mathcal{I} \exists J \in F (I \subseteq J)\}.$$

For $f, g \in {}^{\omega}\omega$, let $f \leq^* g$ if $f(n) \leq g(n)$ for almost all $n \in \omega$.
A family $D \subseteq {}^{\omega}\omega$ is *dominating* if every $f \in {}^{\omega}\omega$ has some $g \in D$ with $f \leq^* g$. A family $B \subseteq {}^{\omega}\omega$ is *unbounded* if no $g \in {}^{\omega}\omega$ exists such that $f \leq^* g$ for all $f \in B$.

The *dominating* and *unbounding* numbers $\mathfrak{d}$ and $\mathfrak{b}$ are the least size of dominating and unbounded families, respectively.

For infinite $X, Y \subseteq \omega$, we say X *splits* or *reaps* Y if both $Y \cap X$ and $Y \setminus X$ are infinite. A family $S \subseteq \mathcal{P}(\omega)$ is *splitting* if every infinite $Y \subseteq \omega$ is split by some $X \in S$. A family $R \subseteq \mathcal{P}(\omega)$ of infinite sets is *unreaped* if no $X \subseteq \omega$ splits all $Y \in R$.

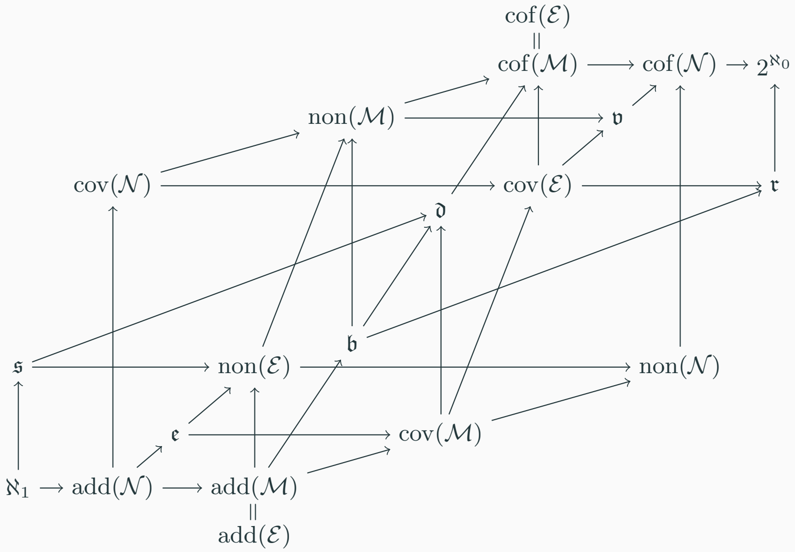
The *splitting* and *reaping* numbers $\mathfrak{s}$ and $\mathfrak{r}$ are the least size of splitting and unreaped families, respectively.

Let $\pi \colon {}^{<\omega}\omega \to \omega$, $f \in {}^{\omega}\omega$ and let $D \subseteq \omega$ be infinite, then we say that π predicts f on D if $\pi(f \restriction n) = f(n)$ for almost all $n \in D$. We call (π, D) a *predictor*.

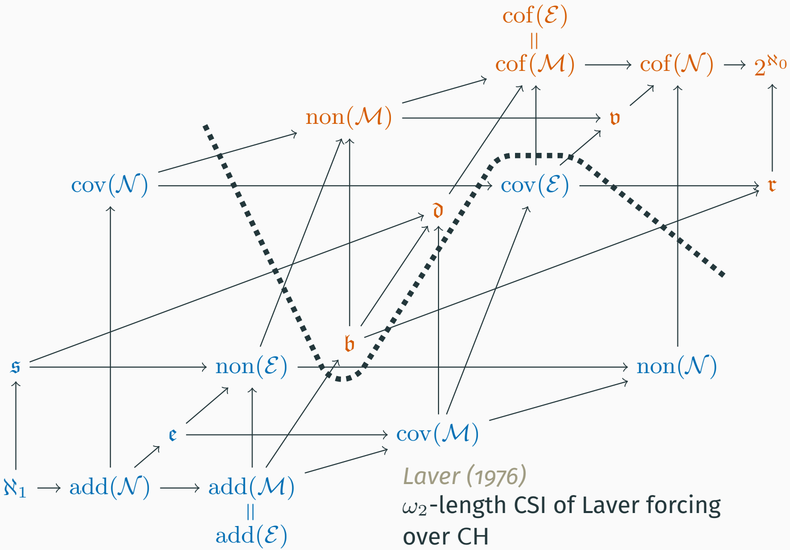
A *predicting* family is a collection P of predictors such that each $f \in {}^{\omega}\omega$ is predicted by some predictor in P. An *evading* family is a family $E \subseteq {}^{\omega}\omega$ such that no predictor predicts every $f \in E$.

The *evasion* and *prediction* numbers $\mathfrak{e}$ and $\mathfrak{v}$ are the least size of evading and predicting families, respectively.

Extended Cichoń Diagram

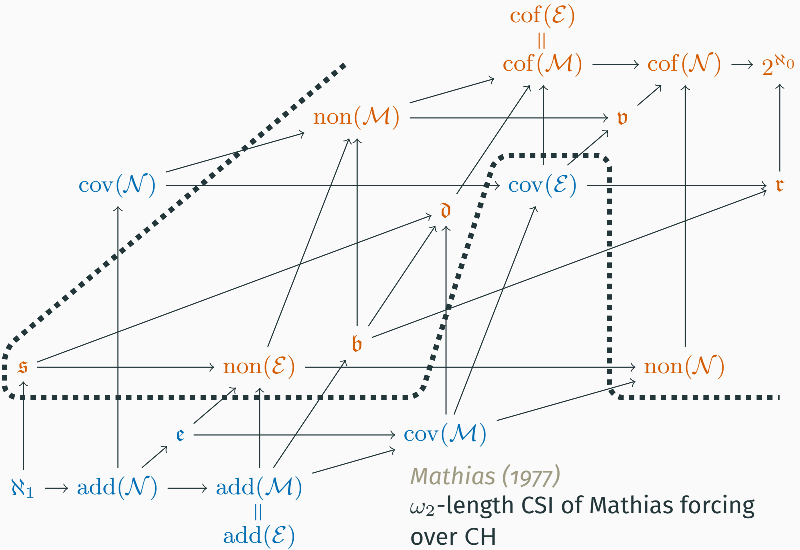


Consistency Results



Laver (1976)
ω_2-length CSI of Laver forcing over CH

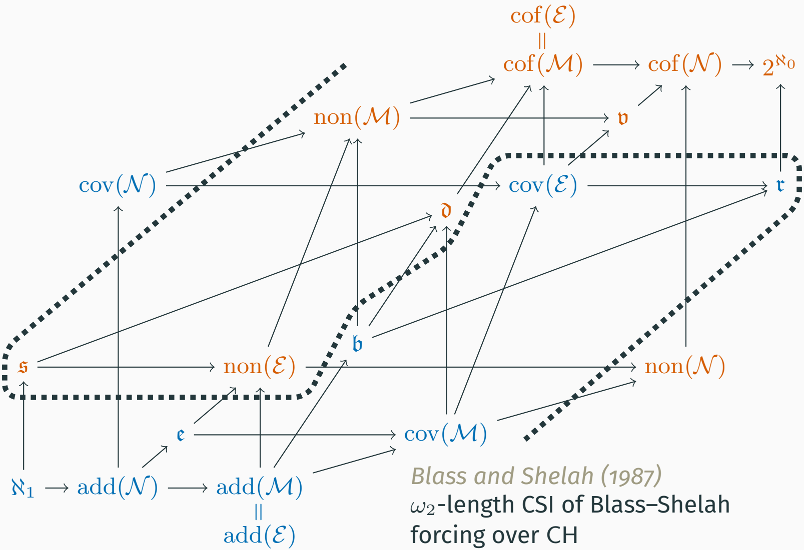
Consistency Results



Mathias (1977)

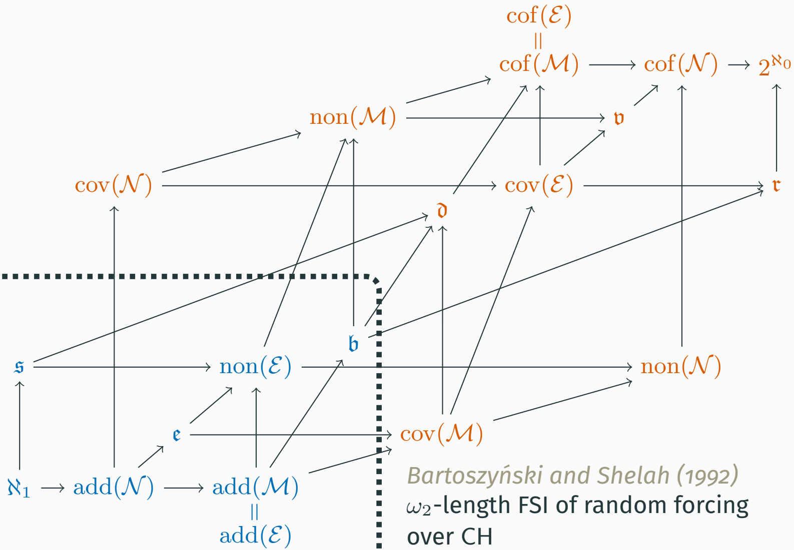
ω_2-length CSI of Mathias forcing over CH

Consistency Results

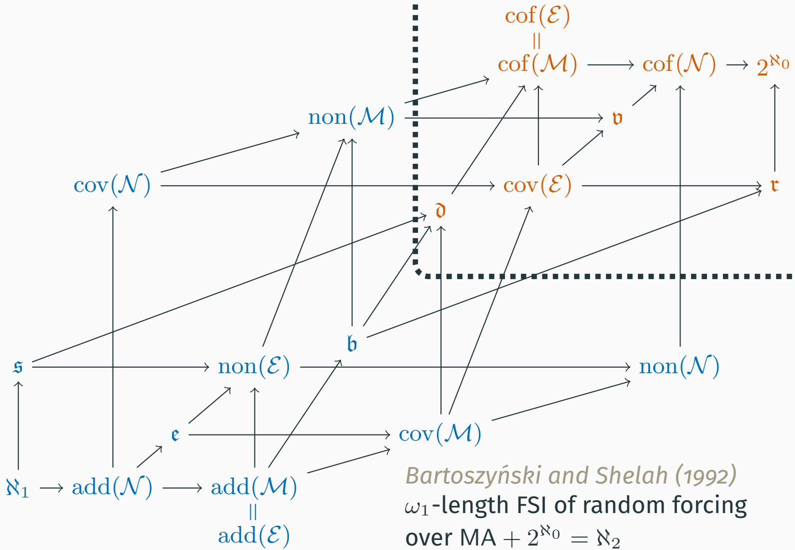


Blass and Shelah (1987)

ω_2-length CSI of Blass–Shelah forcing over CH

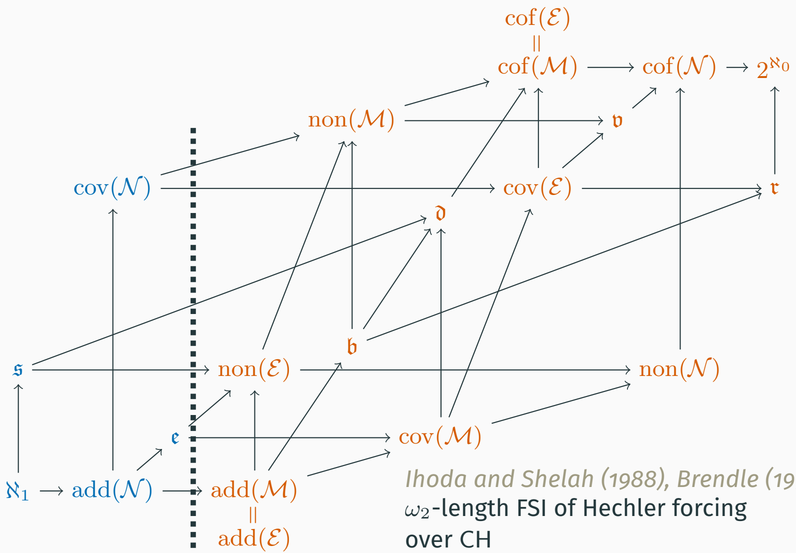


Bartoszyński and Shelah (1992)
ω_2-length FSI of random forcing over CH

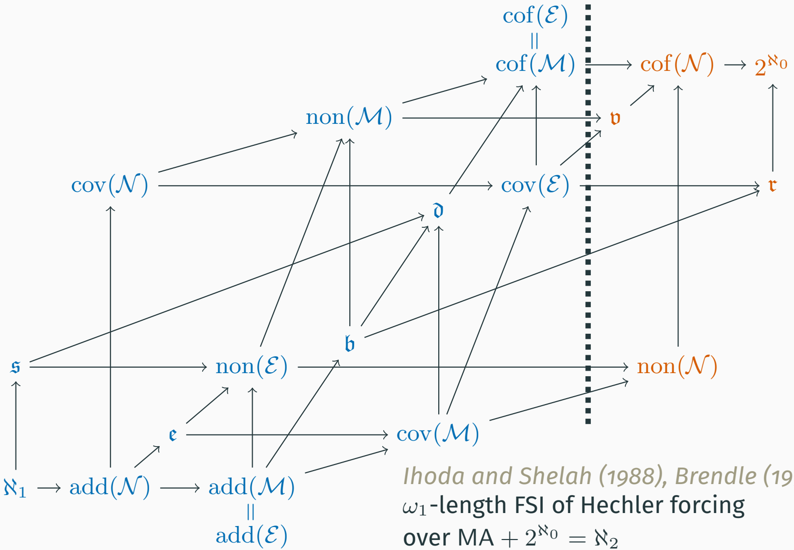


Bartoszyński and Shelah (1992)
ω_1-length FSI of random forcing over MA $+ 2^{\aleph_0} = \aleph_2$

Consistency Results



Ihoda and Shelah (1988), Brendle (1995)
ω_2-length FSI of Hechler forcing
over CH



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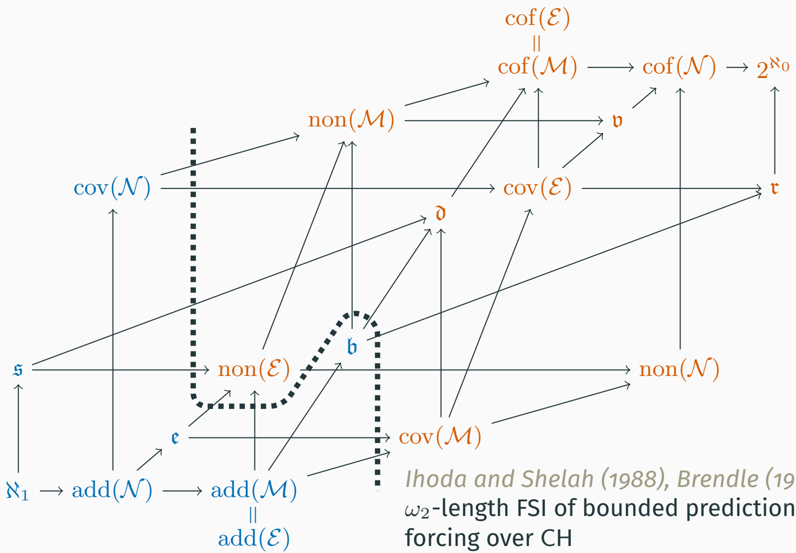
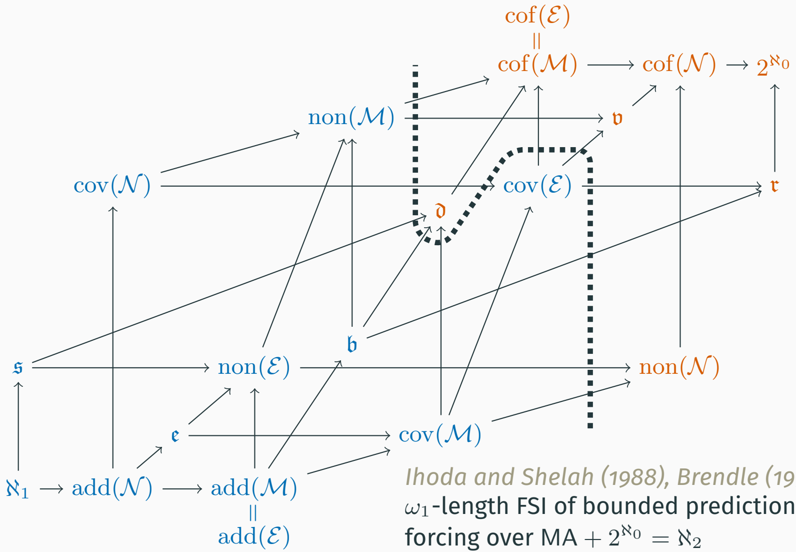
ω_1-length FSI of Hechler forcing
over MA $+ 2^{\aleph_0} = \aleph_2$

Consistency Results



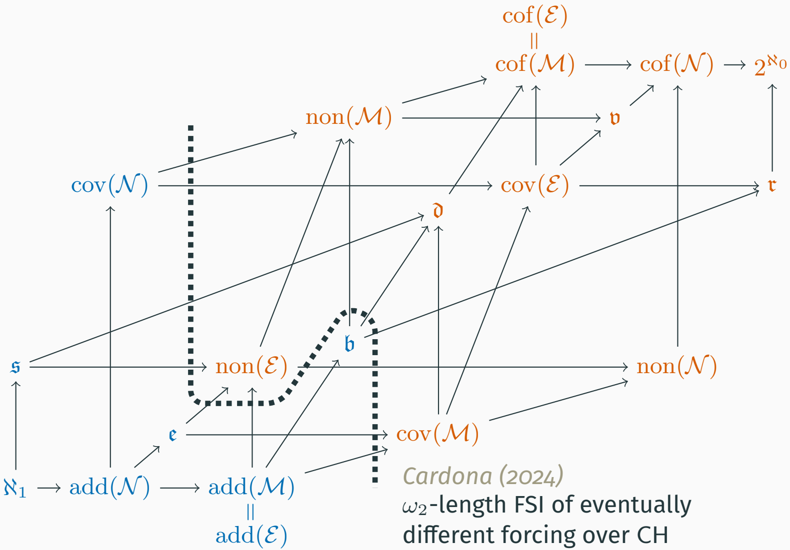
Ihoda and Shelah (1988), Brendle (1995)
ω_2-length FSI of bounded prediction forcing over CH

Consistency Results



Ihoda and Shelah (1988), Brendle (1995)

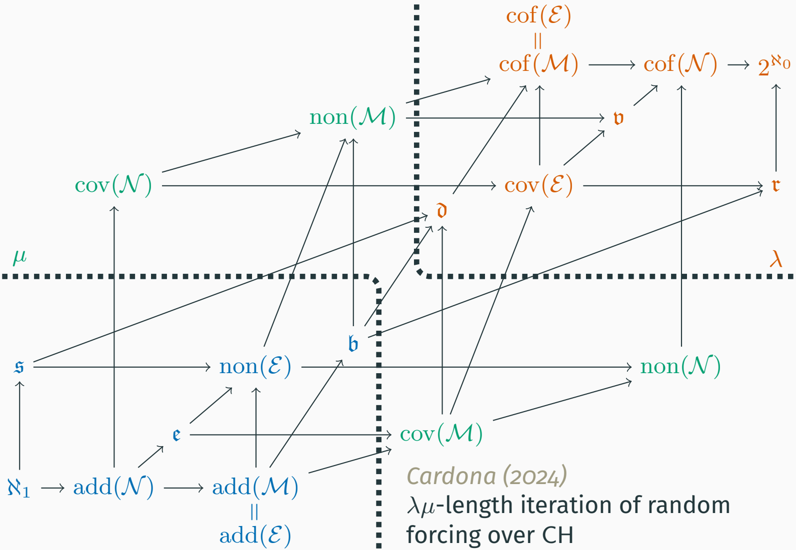
ω_1-length FSI of bounded prediction forcing over MA $+ 2^{\aleph_0} = \aleph_2$



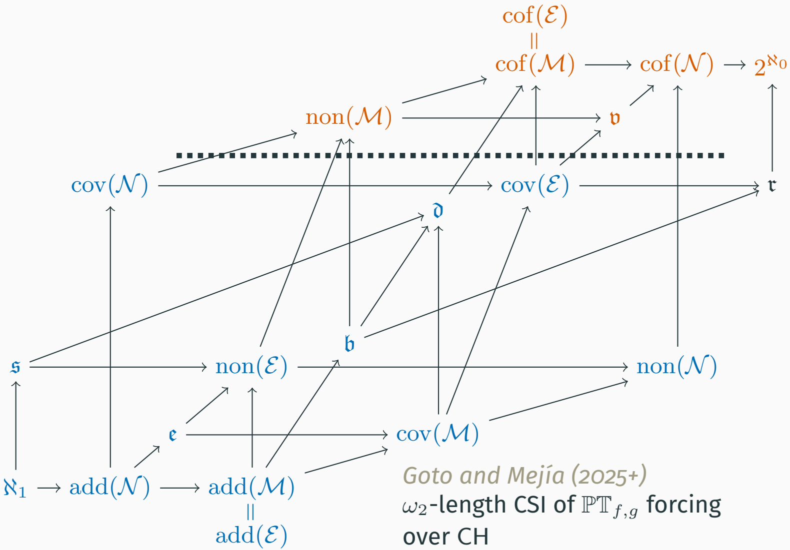
Cardona (2024)

ω_2-length FSI of eventually different forcing over CH

Consistency Results

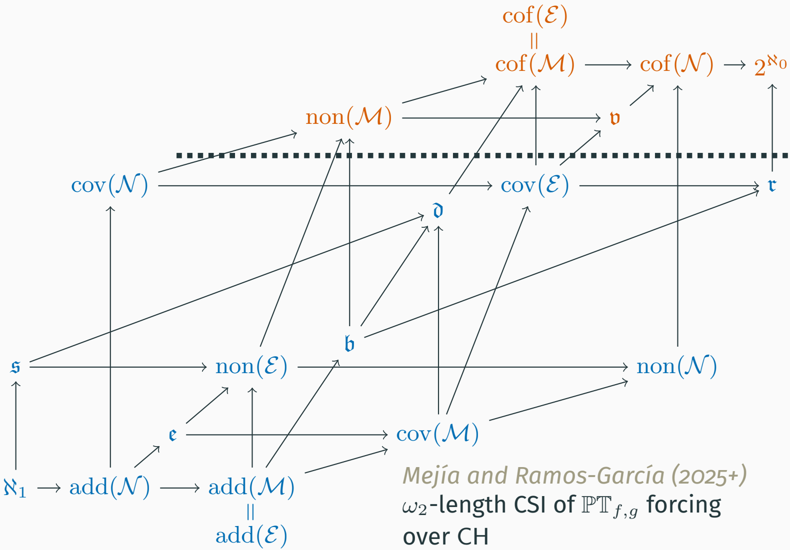


Cardona (2024)
$\lambda\mu$-length iteration of random forcing over CH



Goto and Mejía (2025+)
ω_2-length CSI of $\mathbb{PT}_{f,g}$ forcing over CH

Consistency Results



Mejía and Ramos-García (2025+)
ω_2-length CSI of $\mathbb{PT}_{f,g}$ forcing over CH

Define $\mathfrak{d}_\kappa$, $\mathfrak{b}_\kappa$, $\mathfrak{s}_\kappa$, $\mathfrak{r}_\kappa$, $\mathfrak{e}_\kappa$ and $\mathfrak{v}_\kappa$ by simple replacement of ω by κ. To be precise:

- In the definitions of $\mathfrak{d}_\kappa$ and $\mathfrak{b}_\kappa$, we have $f \leq^* g$ for $f, g \in {}^\kappa\kappa$ if $f(\alpha) \leq g(\alpha)$ for almost all $\alpha \in \kappa$.
- In the definitions of $\mathfrak{s}_\kappa$ and $\mathfrak{r}_\kappa$, we use subsets $X, Y \subseteq \kappa$ of size κ, and X splits Y if $|Y \cap X| = |Y \setminus X| = \kappa$.
- In the definitions of $\mathfrak{e}_\kappa$ and $\mathfrak{v}_\kappa$, the predictor (π, D) consists of $\pi : {}^{<\kappa}\kappa \to \kappa$ and $D \subseteq \kappa$ with $|D| = \kappa$, and $f \in {}^\kappa\kappa$ is predicted by π on D if $f(\alpha) = \pi(f \restriction \alpha)$ holds for almost all $\alpha \in D$.

Theorem *Raghavan and Shelah 2017*

$\mathfrak{s}_\kappa \leq \mathfrak{b}_\kappa$, and if $\beth_\omega < \kappa$, then $\mathfrak{d}_\kappa \leq \mathfrak{r}_\kappa$.

Theorem *Zapletal 1997, Ben-Neria and Gitik 2015*

$\kappa^+ \leq \mathfrak{s}_\kappa$ if and only if κ is weakly compact.

$\kappa^{+n} \leq \mathfrak{s}_\kappa$ implies the existence of an inner model in which κ is measurable with Mitchell order $o(\kappa) = \kappa^{+n}$.

Let $\varphi \in \prod_{\alpha\in\kappa}[\kappa]^{|\alpha|}$ and $f \in {}^\kappa\kappa$, then we say φ *localises* f if $f(\alpha) \in \varphi(\alpha)$ for almost all $\alpha \in \kappa$.

A family $L \subseteq \prod_{\alpha\in\kappa}[\kappa]^{|\alpha|}$ is called *localising* if every $f \in {}^\kappa\kappa$ is localised by some $\varphi \in L$, and let us call a family $U \subseteq {}^\kappa\kappa$ *unlocalisable* if no $\varphi \in \prod_{\alpha\in\kappa}[\kappa]^{|\alpha|}$ localises all $f \in U$.

The *localisation* and *unlocalisation* numbers $\mathfrak{d}_\kappa(\in^*)$ and $\mathfrak{b}_\kappa(\in^*)$ are the least size of localising and unlocalisable families, respectively.

Theorem *Bartoszyński 1987*

$\mathrm{cof}(\mathcal{N}) = \mathfrak{d}(\in^*)$ and $\mathrm{add}(\mathcal{N}) = \mathfrak{b}(\in^*)$.

Extended Cichoń Diagram (Without $\mathcal{N}$ and $\mathcal{E}$)

$$
\begin{tikzcd}
 & & & & & & \operatorname{cof}(\mathcal{M}) \arrow[rr] & & \mathfrak{d}(\in^*) \arrow[r] & 2^{\aleph_0} \\
 & & & & \operatorname{non}(\mathcal{M}) \arrow[urr] \arrow[rrr] & & & \mathfrak{v} \arrow[ur] & & \\
 & & & & & \mathfrak{d} \arrow[uur] & & & & \mathfrak{r} \arrow[uu] \\
\mathfrak{s} \arrow[uurrrr] \arrow[urrrrr] & & & & \mathfrak{b} \arrow[uu] \arrow[ur] \arrow[urrrrr] & & & & & \\
 & & \mathfrak{e} \arrow[rrr] \arrow[uuuurr] & & & \operatorname{cov}(\mathcal{M}) \arrow[uu] \arrow[uuurr] \arrow[uurrrr] & & & & \\
\aleph_1 \arrow[r] \arrow[uu] & \mathfrak{b}(\in^*) \arrow[rr] \arrow[ur] & & \operatorname{add}(\mathcal{M}) \arrow[urr] \arrow[uur] & & & & & &
\end{tikzcd}
$$

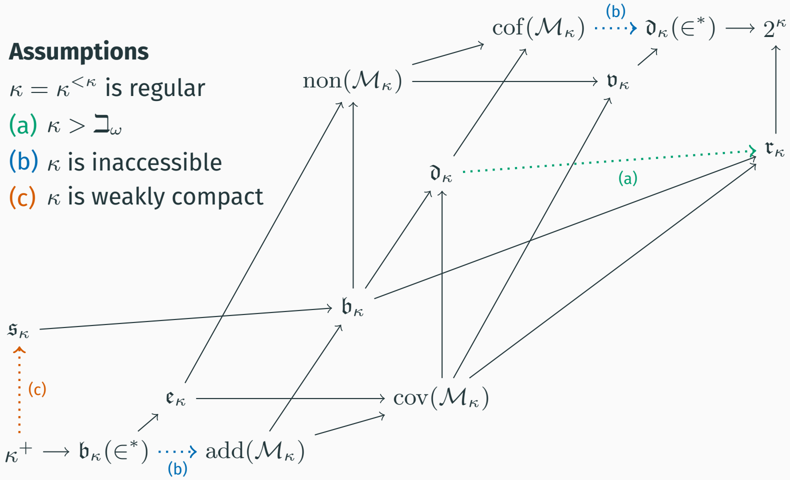
Assumptions

$\kappa = \kappa^{<\kappa}$ is regular

(a) $\kappa > \beth_\omega$

(b) κ is inaccessible

(c) κ is weakly compact



Classically we have the following results for $\mathrm{non}(\mathcal{E})$:

- $\mathrm{non}(\mathcal{E}) \leq \mathrm{non}(\mathcal{M})$,
- $\mathrm{add}(\mathcal{M}) \leq \mathrm{non}(\mathcal{E})$,
- $\mathfrak{e} \leq \mathrm{non}(\mathcal{E})$,
- $\mathfrak{s} \leq \mathrm{non}(\mathcal{E})$,

and the following results for $\mathrm{add}(\mathcal{E})$:

- $\mathrm{add}(\mathcal{M}) = \mathrm{add}(\mathcal{E})$,
- $\mathrm{add}(\mathcal{E}) \leq \mathrm{cov}(\mathcal{M})$,
- $\mathrm{add}(\mathcal{E}) \leq \mathfrak{b}$,
- $\mathfrak{b}(\in^*) \leq \mathrm{add}(\mathcal{E})$.

The question is which of these generalise.

Work in Progress for the Higher Relations

So far, we have:

✓ $\mathrm{non}(\mathcal{E}_\kappa) \leq \mathrm{non}(\mathcal{BE}_\kappa) \leq \mathrm{non}(\mathcal{M}_\kappa)$,

? $\mathrm{add}(\mathcal{M}_\kappa) \leq \mathrm{non}(\mathcal{E}_\kappa)$,

✓ $\mathfrak{e}_\kappa \leq \mathrm{non}(\mathcal{BE}_\kappa)$, ? $\mathfrak{e}_\kappa \leq \mathrm{non}(\mathcal{E}_\kappa)$,

✓ $\mathfrak{s}_\kappa \leq \mathrm{non}(\mathcal{BE}_\kappa)$, ? $\mathfrak{s}_\kappa \leq \mathrm{non}(\mathcal{E}_\kappa)$,

and:

?? $\mathrm{add}(\mathcal{M}_\kappa) = \mathrm{add}(\mathcal{E}_\kappa)$ (due to Lebesgue measure),

? $\mathrm{add}(\mathcal{E}_\kappa) \leq \mathrm{cov}(\mathcal{M}_\kappa)$,

? $\mathrm{add}(\mathcal{E}_\kappa) \leq \mathfrak{b}_\kappa$, ✓ $\mathrm{add}(\mathcal{E}_\kappa^-) \leq \mathfrak{b}_\kappa$, ? $\mathfrak{d}_\kappa \leq \mathrm{cof}(\mathcal{E}_\kappa^-)$,

? $\mathfrak{b}_\kappa(\in^*) \leq \mathrm{add}(\mathcal{E}_\kappa)$.

In conclusion, many questions remain.

Theorem *Marton, Šupina, Repický, and vdV. 2025+*

$\mathfrak{s}_\kappa \leq \mathrm{non}(\mathcal{BE}_\kappa)$.

Proof. If $A \subseteq {}^\kappa 2$ and $A \notin \mathcal{BE}_\kappa$, then $\{x^{-1}(1) \mid x \in A\}$ is a splitting family.

Let $Y \in [\kappa]^\kappa$ and without loss we assume $|\kappa \setminus Y| = \kappa$, then we define $B \in \mathrm{BP}_\kappa$ such that $|B_\alpha \cap Y| = 2^{|\alpha|}$ and $|B_\alpha \setminus Y| = |\alpha|$. Define $\varphi \in \Sigma_\kappa(B)$ by

$$\varphi(\alpha) = \{s \in {}^{B_\alpha}2 \mid s \restriction (B_\alpha \cap Y) \text{ is constant}\}$$

Since $A \notin \mathcal{BE}_\kappa$, there is $x \in A$ with $x \notin [\varphi]_*$. Hence, for cofinally many $\alpha \in \kappa$ we see that $x \restriction (B_\alpha \cap Y)$ is not constant. It follows that $|\{x(\xi) = i \mid \xi \in Y\}| = \kappa$ for both $i \in 2$, thus $x^{-1}(1)$ splits Y. $\square$

For $f, g \in {}^\kappa\kappa$, define $f \leq^{\mathrm{cl}} g$ if there is a club set $C \subseteq \kappa$ such that $f(\alpha) \leq g(\alpha)$ for all $\alpha \in C$. We define $\mathfrak{b}_\kappa^{\mathrm{cl}}$ as the least size of a $\leq^{\mathrm{cl}}$-unbounded family.

Theorem *Cummings and Shelah 1995, Theorem 6*

$\mathfrak{b}_\kappa^{\mathrm{cl}} = \mathfrak{b}_\kappa$.

We may dually describe $\mathfrak{d}_\kappa^{\mathrm{cl}}$ as the least size of a $\leq^{\mathrm{cl}}$-dominating family. There is the following long-standing open question:

Theorem *Cummings and Shelah 1995*

Is $\mathfrak{d}_\kappa^{\mathrm{cl}} = \mathfrak{d}_\kappa$?

Theorem *Cummings and Shelah 1995, Theorem 8*

If $\kappa > \beth_\omega$, then $\mathfrak{d}_\kappa^{\mathrm{cl}} = \mathfrak{d}_\kappa$.

Theorem *Marton, Šupina, Repický, and vdV. 2025+*

$\mathrm{add}(\mathcal{E}_\kappa^-) \leq \mathfrak{b}_\kappa^{\mathrm{cl}}$.

Proof. Given an increasing $b \in {}^\kappa\kappa$, let $I^b = \langle I^b_\alpha \mid \alpha \in \kappa \rangle \in \mathrm{IP}_\kappa$ be defined by $\min(I^b_\alpha) = i^b_\alpha$ and $i^b_{\alpha+1} = i^b_\alpha + 2^{|b(\alpha)|}$. Choose $\varphi^b \in \Sigma_\kappa(I^b)$ such that for any $s \colon [i^b_\alpha, i^b_\alpha + \gamma) \to 2$ and $\gamma \leq b(\alpha)$ there exists $t_s \in \varphi^b(\alpha)$ with $s \subseteq t_s$. We claim that if $B \subseteq {}^\kappa\kappa$ is $\leq^{\mathrm{cl}}$-unbounded, then $X = \bigcup_{b \in B} [\varphi^b]_* \notin \mathcal{E}_\kappa^-$.

Consider $[\psi]_*$ for $\psi \in \Sigma_\kappa(J)$ and define $f \colon \alpha \mapsto \mathrm{ot}(J_\alpha)$. Let $b \in B$ be such that $b \not\leq^{\mathrm{cl}} f$ and let $\alpha \in S$ iff $f(\alpha) \leq b(\alpha)$ and $i^b_\alpha = \min(J_\alpha)$, then S is stationary. For each $\alpha \in S$ we choose $s_\alpha \in {}^{J_\alpha}2 \setminus \psi(\alpha)$, then $t_{s_\alpha} \in {}^{I^b_\alpha}2 \cap \varphi^b(\alpha)$. Thus, there is $x \in X$ with $x \restriction I^b_\alpha = t_{\tilde{s}_\alpha}$ for all $\alpha \in S$, and consequently $x \notin [\psi]_*$. $\square$

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