Dominating and Eventually Different κ-reals

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How do you separate cardinal characteristics?

Answer: (Usually) to force $\lambda = \mathfrak{x} < \mathfrak{y} = \kappa$, your strategies are:

(1) Start with a model where $\mathfrak{x} = \mathfrak{y} = \lambda$ and add **witnesses** for $\mathfrak{y} = \kappa$ without disturbing witnesses for $\mathfrak{x} = \lambda$

(2) Start with a model where $\mathfrak{x} = \mathfrak{y} = \kappa$ and add witnesses for $\mathfrak{x} = \lambda$ without disturbing witnesses for $\mathfrak{y} = \kappa$

For instance, we can force $\mathfrak{b} < \mathfrak{d} \cdots$

$\cdots$ by adding $\aleph_2$-many *Cohen reals* over $\mathbf{V} \models$ " $\mathfrak{b} = \mathfrak{d} = \aleph_1$ ", or

$\cdots$ by adding $\aleph_1$-many *Cohen reals* over $\mathbf{V} \models$ " $\mathfrak{b} = \mathfrak{d} = \aleph_2$ "

Question: Which forcing notions add which kinds of witnesses?

We will assume that κ is an **inaccessible** cardinal. The **generalised Baire space** ${}^{\kappa}\kappa$ is the set of functions $f : \kappa \to \kappa$, called κ-**reals**.

Given functions $f, f' \in {}^{\kappa}\kappa$ and a relation $\lhd \subseteq \kappa \times \kappa$, we write

$$\begin{aligned} f \lhd f' &\Leftrightarrow \forall \alpha \in \kappa (f(\alpha) \lhd f'(\alpha)), \\ f \lhd^* f' &\Leftrightarrow \exists \alpha_0 \in \kappa \forall \alpha \geq \alpha_0 (f(\alpha) \lhd f'(\alpha)), \\ f \lhd^\infty f' &\Leftrightarrow \forall \alpha_0 \in \kappa \exists \alpha \geq \alpha_0 (f(\alpha) \lhd f'(\alpha)). \end{aligned}$$

$$f \not\lhd f' \Leftrightarrow \neg(f \lhd f')$$
$$f \not\lhd^* f' \Leftrightarrow \neg(f \lhd^* f')$$
$$f \not\lhd^\infty f' \Leftrightarrow \neg(f \lhd^\infty f')$$

Let $\mathbf{V} \subseteq \mathbf{W}$ be models of ZFC. We call a κ-real $f \in ({}^{\kappa}\kappa)^{\mathbf{W}}$ ...

- ... **dominating** over $\mathbf{V}$ if $g \leq^* f$ for all $g \in ({}^{\kappa}\kappa)^{\mathbf{V}}$.
- ... **unbounded** over $\mathbf{V}$ if $f \not\leq^* g$ for all $g \in ({}^{\kappa}\kappa)^{\mathbf{V}}$.
- ... **eventually different** over $\mathbf{V}$ if $f \neq^\infty g$ for all $g \in ({}^{\kappa}\kappa)^{\mathbf{V}}$.
- ... **unbounded non-dominating eventually different (unded)** over $\mathbf{V}$ if f is eventually different and unbounded over $\mathbf{V}$, but not dominating.

Let $\mathbf{V} \subseteq \mathbf{W}$ be models of ZFC with $b \in ({}^{\kappa}\kappa)^{\mathbf{V}}$. We assume $b(\alpha)$ is an infinite cardinal for all $\alpha \in \kappa$. Define:

$$\prod b = \prod_{\alpha\in\kappa} b(\alpha) = \{f \in {}^{\kappa}\kappa \mid f < b\}.$$

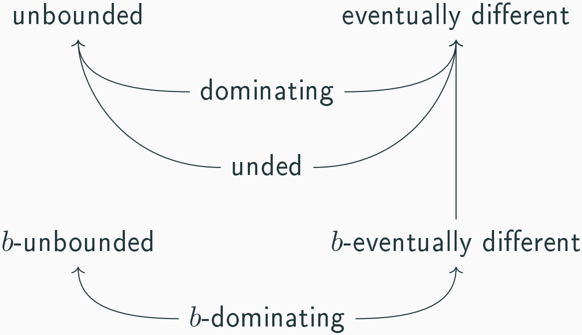
We call a bounded κ-real $f \in (\prod b)^{\mathbf{W}}$ ...

... b-**dominating** over $\mathbf{V}$ if $g \leq^* f$ for all $g \in (\prod b)^{\mathbf{V}}$.

... b-**unbounded** over $\mathbf{V}$ if $f \not\leq^* g$ for all $g \in (\prod b)^{\mathbf{V}}$.

... b-**eventually different** over $\mathbf{V}$ if $f \neq^\infty g$ for all $g \in (\prod b)^{\mathbf{V}}$.

Some Simple Observations



An arrow $P \to Q$ means that the existence of a κ-real with property P over $\mathbf{V}$ implies the existence of a κ-real with property Q over $\mathbf{V}$.

Question 1: Is the diagram complete?

Question 2: Which forcing notions add which kinds of κ-reals?

Forcing Notions

We will look at forcing notions that preserve ${}^{<\kappa}\kappa$ (that is, are $<\kappa$-distributive), preserve cardinals, and add a new κ-real:

- κ-Cohen forcing $\mathbb{C}_\kappa$
- κ-Hechler forcing $\mathbb{D}_\kappa$
- Bounded κ-Hechler forcing $\mathbb{D}^b_\kappa$
- κ-Eventually Different forcing $\mathbb{E}_\kappa$
- κ-Laver forcing guided by a filter $\mathbb{L}^{\mathcal{U}}_\kappa$
- κ-Miller forcing guided by an filter $\mathbb{M}\mathbb{i}^{\mathcal{U}}_\kappa$
- Bounded κ-Miller forcing (κ-Miller Lite forcing) $\mathbb{ML}^b_\kappa$

A forcing notion $\mathbb{P}$ is $<\kappa$-**closed** if for every descending sequence of conditions of length $<\kappa$ has a lower bound in $\mathbb{P}$.

$\mathcal{G}(\mathbb{P}, p)$ denotes a game of length κ, where at stage $\alpha \in \kappa$, White chooses a condition p_α stronger than all previous Black moves and Black subsequently chooses $p'_\alpha \leq p_\alpha$. White wins $\mathcal{G}(\mathbb{P}, p)$ if White can make moves at every stage $\alpha \in \kappa$. A forcing $\mathbb{P}$ is **strategically** $<\kappa$-**closed** if White has a winning strategy for $\mathcal{G}(\mathbb{P}, p)$ for all $p \in \mathbb{P}$.

A forcing $\mathbb{P}$ is $<\kappa$-**distributive** if for any sequence $\langle D_\alpha \mid \alpha \in \lambda \rangle$ with $\lambda < \kappa$ and each $D_\alpha \subseteq \mathbb{P}$ open dense, also $\bigcap_{\alpha \in \lambda} D_\alpha$ is dense.

We have the following implications:

$<\kappa$-closed $\Rightarrow$ strategically $<\kappa$-closed $\Rightarrow$ $<\kappa$-distributive

A $<\kappa$-distributive forcing notion $\mathbb{P}$ preserves all cardinals $\leq\kappa$.

A forcing $\mathbb{P}$ is $<\mu$-**c.c.** if all antichains are of size $<\mu$. If $\mathbb{P}$ is $<\mu$-c.c., it preserves all cardinals $\geq\mu$.

We say $A \subseteq \mathbb{P}$ is $<\lambda$-**linked** if every $B \in [A]^{<\lambda}$ has a lower bound (in $\mathbb{P}$). We call $\mathbb{P}$ (μ, λ)-**centred** if $\mathbb{P}$ is a μ-union of $<\lambda$-linked sets.

We say $A \subseteq \mathbb{P}$ has **calibre** λ if for every $B \in [A]^{\lambda}$ there exists $q \in \mathbb{P}$ such that $|\{p \in B \mid q \leq p\}| = \lambda$. We say $\mathbb{P}$ is (μ, λ)-**calibre** if it is a μ-union of λ-calibre sets.

If $\mathbb{P}$ is (μ, λ)-centred or (μ, λ)-calibre for any $3 \leq \lambda \leq \mu$, then $\mathbb{P}$ is $<\mu^+$-c.c., and thus $\mathbb{P}$ preserves cardinals $\geq\mu^+$.

The κ-**Cohen forcing** $\mathbb{C}_\kappa$ has conditions $s \in {}^{<\kappa}\kappa$. The ordering is defined by $t \leq s$ iff $s \subseteq t$. $\mathbb{C}_\kappa$ is $<\kappa$-closed and (trivially) $<\kappa^+$-c.c.. $\mathbb{C}_\kappa$ adds a κ-**Cohen real** $\bigcup G \in {}^\kappa\kappa$, where G is a $\mathbb{C}_\kappa$-generic filter.

A κ-Cohen real is unbounded over $\mathbf{V}$.

Consider $f_b \in \prod b$, where $f = \bigcup G$ is a κ-Cohen real and $f_b(\alpha)$ is such that there exists $\beta \in \kappa$ with $f(\alpha) = b(\alpha) \cdot \beta + f_b(\alpha)$. Then f_b is b-unbounded.

But does $\mathbf{V}^{\mathbb{C}_\kappa}$ contain any other kind of κ-real?

Lemma *Similar to [Bartoszyński and Judah, 1995, Lemma 3.1.2] for ${}^\omega\omega$*

$\mathbb{C}_\kappa$ does not add any eventually different κ-reals.

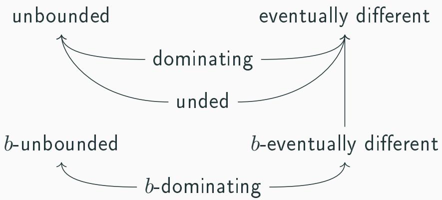
Proof. Since $2^{<\kappa} = \kappa$, we can enumerate $\mathbb{C}_\kappa$ as $\{p_\alpha \mid \alpha \in \kappa\}$.
Suppose $\dot{f}$ is a name for an eventually different κ-real over $\mathbf{V}$.

Define $g(\alpha) = \min\left\{\xi \in \kappa \,\middle|\, p_\alpha \nVdash \text{“} \dot{f}(\alpha) \neq \xi \text{”}\right\}$.

Now suppose that $p \in \mathbb{C}_\kappa$ and $p \Vdash \text{“} \dot{f} \neq^\infty g \text{”}$, then there is some α_0 and $p' \leq p$ such that $p' \Vdash \text{“} \dot{f}(\alpha) \neq g(\alpha) \text{”}$ for all $\alpha \geq \alpha_0$.

But then there is $\alpha \geq \alpha_0$ such that $p_\alpha \leq p'$ and $p_\alpha \nVdash \text{“} \dot{f}(\alpha) \neq g(\alpha) \text{”}$, contradiction. □

Overview



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{Mi}^{\mathcal{U}}_\kappa$	$\mathbb{ML}^b_\kappa$
unbounded	✓						
eventually different	✗						
dominating	✗						
unded	✗						
b-unbounded	✓						
b-eventually different	✗						
b-dominating	✗						

The κ-**Hechler forcing** $\mathbb{D}_\kappa$ has conditions (s, f) where $s \in {}^{<\kappa}\kappa$ and $f \in {}^{\kappa}\kappa$. The ordering is defined by $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $f(\alpha) \leq g(\alpha)$ for $\alpha \in \kappa \setminus \mathrm{dom}(t)$ and $f(\alpha) \leq t(\alpha)$ for $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

$\mathbb{D}_\kappa$ is $<\kappa$-closed and (κ, κ)-centred (since the subsets $D_s = \{(t, f) \in \mathbb{D}_\kappa \mid t = s\}$ for $s \in {}^{<\kappa}\kappa$ are $<\kappa$-linked).

$\mathbb{D}_\kappa$ adds a κ-**Hechler real** $\bigcup\{s \mid (s, \cdot) \in G\}$, where G is $\mathbb{D}_\kappa$-generic. A κ-Hechler real f is dominating over $\mathbf{V}$, since $(s, g) \Vdash$ "$g \leq^* \dot{f}$". Moreover, $\mathbb{D}_\kappa$ adds a κ-Cohen real, and hence a b-unbounded κ-real over $\mathbf{V}$.

Lemma *Similar to [Bartoszyński and Judah, 1995, Lemma 6.5.30] for ${}^{\omega}\omega$*

If $\mathbb{P}$ is (κ, κ)-centred, it does not add b-eventually different κ-reals.

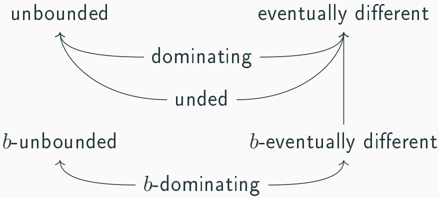
Proof. Let $\mathbb{P} = \bigcup_{\gamma\in\kappa} P_\gamma$ such that each P_γ is $<\kappa$-linked, and let $\Vdash_{\mathbb{P}}$ " $\dot{f} \in \prod b$".

Define $g_\gamma(\alpha) = \min\left\{\xi \in b(\alpha) \mid \forall p \in P_\gamma(p \nVdash \text{" } \dot{f}(\alpha) \neq \xi)\text{"}\right\}$, then $g_\gamma(\alpha) \in b(\alpha)$ (If not, for each $\beta \in b(\alpha)$ find $p_\beta \in P_\gamma$ with $p_\beta \Vdash$ " $\dot{f}(\alpha) \neq \beta$", then $\{p_\beta \mid \beta \in b(\alpha)\}$ has no common extension.)

Suppose $h =^\infty g_\gamma$ for all $\gamma \in \kappa$. If $\alpha_0 \in \kappa$ and $p \in P_\gamma$, then we can find $\alpha \geq \alpha_0$ such that $h(\alpha) = g_\gamma(\alpha)$. But, we know that $p \nVdash$ " $\dot{f}(\alpha) \neq g_\gamma(\alpha)$". Therefore $p' \Vdash$ " $\dot{f}(\alpha) = h(\alpha)$" for some $p' \leq p$. Since α_0 and p were arbitrary, we see that $\Vdash_{\mathbb{P}}$ " $\dot{f} =^\infty h$".

Thus $\dot{f}$ does not name a b-eventually different κ-real. $\square$

Overview



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{Mi}^{\mathcal{U}}_\kappa$	$\mathbb{ML}^b_\kappa$
unbounded	✓	✓					
eventually different	✗	✓					
dominating	✗	✓					
unded	✗	?					
b-unbounded	✓	✓					
b-eventually different	✗	✗					
b-dominating	✗	✗					

Let $b \in {}^{\kappa}\kappa$ be increasing and $\mathrm{cf}(b(\alpha)) > \bigcup_{\xi<\alpha} b(\xi)$ for limit α.

The b-κ-**Hechler forcing** $\mathbb{D}^b_\kappa$ has conditions (s, f) where $s \in \prod_{<\kappa} b$ and $f \in \prod b$. The ordering is defined by $(t, g) \leq (s, f)$ iff $s \subseteq t$ and $f(\alpha) \leq g(\alpha)$ for $\alpha \in \kappa \setminus \mathrm{dom}(t)$ and $f(\alpha) \leq t(\alpha)$ for $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

$\mathbb{D}^b_\kappa$ is strategically $<\kappa$-closed and has a (κ, λ)-centred dense subset for each $\lambda < \kappa$.

$\mathbb{D}^b_\kappa$ adds a b-κ-**Hechler real** $\bigcup\{s \mid (s, \cdot) \in G\}$ to $\prod b$, where G is $\mathbb{D}^b_\kappa$-generic. A b-κ-Hechler real is b-dominating over $\mathbf{V}$, hence $\mathbb{D}^b_\kappa$ is not (κ, κ)-centred. Moreover, $\mathbb{D}^b_\kappa$ adds a κ-Cohen real and hence an unbounded κ-real as well.

Lemma *Brendle, private communication*

If $\mathbb{P}$ is (κ, κ)-calibre, then it does not add a dominating κ-real.

Proof. Let $\mathbb{P} = \bigcup_{\gamma \in \kappa} P_\gamma$ with all P_γ of calibre κ and $\Vdash_{\mathbb{P}}$ " $\dot{f} \in {}^\kappa\kappa$ ". We define $g_\gamma(\alpha) = \min \left\{ \xi \in \kappa \mid \forall p \in P_\gamma (p \nVdash \text{" } \dot{f}(\alpha) > \xi \text{ "}) \right\}$, then $g_\gamma(\alpha) \in \kappa$. (If not, then for each $\beta \in \kappa$ there is some $p_\beta \in P_\gamma$ with $p_\beta \Vdash$ " $\dot{f}(\alpha) > \beta$ ". Since P_γ has calibre κ, there is some $q \in \mathbb{P}$ with $q \leq p_\beta$ for κ-many $\beta \in \kappa$, contradiction.)

Find $h \in {}^\kappa\kappa$ with $g_\gamma <^* h$ for all $\gamma \in \kappa$ and let α_γ be such that $g_\gamma(\alpha) < h(\alpha)$ for all $\alpha \geq \alpha_\gamma$. For each $p \in P_\gamma$ and $\alpha \geq \alpha_\gamma$ we have $p \nVdash$ " $\dot{f}(\alpha) > g_\gamma(\alpha)$ ". Hence there exists $p' \leq p$ such that $p' \Vdash$ " $\dot{f}(\alpha) \leq g_\gamma(\alpha) < h(\alpha)$ ". Therefore $\Vdash_{\mathbb{P}}$ " $h \not\leq^* \dot{f}$ " and $\dot{f}$ does not name a dominating κ-real. $\square$

Lemma *Brendle, private communication*

If κ is weakly compact, $\mathbb{D}^b_\kappa$ has (κ,κ)-calibre.

Proof. For any $s \in \prod_{<\kappa} b$ and $\{f_\alpha \mid \alpha \in \kappa\} \subseteq \prod b$, we find some $f \in \prod b$ and $A \in [\kappa]^\kappa$ such that $f(\xi) \geq f_\alpha(\xi)$ for all $\xi \in \kappa \setminus \mathrm{dom}(s)$ and $\alpha \in A$. Then $(s, f) \leq (s, f_\alpha)$ for all $\alpha \in A$, hence $\mathbb{D}^b_\kappa = \bigcup_{s \in \prod_{<\kappa} b} \{s\} \times \prod b$ has (κ,κ)-calibre.

W.l.o.g. $f_\alpha \neq f_\beta$ and $s \subseteq f_\alpha$ for all $\alpha < \beta \in \kappa$. Let $g \in \prod b$ be such that $s \subseteq g$ and $f_\alpha \leq^* g$ for all $\alpha \in \kappa$.

Define $T = \left\{t \in \prod_{<\kappa} b \mid \exists\alpha\exists\beta(\alpha \neq \beta \wedge t \subseteq f_\alpha \cap f_\beta)\right\}$.

T is a κ-tree and has a cofinal branch $g' \in [T]$. For any $\alpha_0, \gamma \in \kappa$ there exists $\alpha \geq \alpha_0$ such that $\gamma \subseteq \mathrm{dom}(f_\alpha \cap g')$. $\cdots$

$\cdots$ We construct $f \in \prod b$ by recursion. Let $\gamma_0 = \gamma_0^* = \mathrm{dom}(s)$ and α_0 be arbitrary and $f \restriction \gamma_0 = s$. Given γ_η, α_η and $f \restriction \gamma_\eta$ for all $\eta < \xi$, let $\gamma_\xi^* = \sup_{\eta<\xi} \gamma_\eta$. We choose some $\alpha_\xi > \alpha_\eta$ for all $\eta < \xi$ such that $\gamma_\xi^* + 1 \subseteq \mathrm{dom}(f_{\alpha_\xi} \cap g')$ and we let $\gamma_\xi > \gamma_\xi^*$ be such that $f_{\alpha_\xi}(\beta) \leq g(\beta)$ for all $\beta \geq \gamma_\xi$.

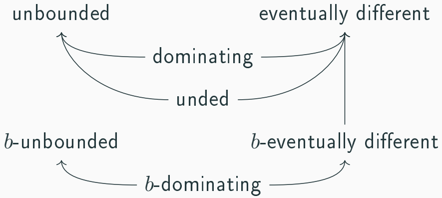
Let $f \restriction [\gamma_\xi^*, \gamma_\xi) : \beta \mapsto \max\left\{f_{\alpha_\xi}(\beta), g(\beta), g'(\beta)\right\}$, then $f_{\alpha_\xi} \leq f$.

Let $\xi \in \kappa$ and consider the following four cases:

If $\beta \in \mathrm{dom}(s)$, then $f_{\alpha_\xi}(\beta) = s(\beta) = f(\beta)$.
If $\beta \in [\mathrm{dom}(s), \gamma_\xi^*) = \sup_{\eta<\xi} \gamma_\eta$, then $f_{\alpha_\xi}(\beta) = g'(\beta) \leq f(\beta)$.
If $\beta \in [\gamma_\xi^*, \gamma_\xi)$, then $f_{\alpha_\xi}(\beta) \leq f(\beta)$ by definition.
If $\beta \in [\gamma_\xi, \kappa)$, then $f_{\alpha_\xi}(\beta) \leq g(\beta) \leq f(\beta)$.

Therefore $f \geq f_{\alpha_\xi}$. We define $A = \{\alpha_\xi \mid \xi \in \kappa\}$, then $|A| = \kappa$ and $(s, f) \leq (s, f_\alpha)$ for all $\alpha \in A$. $\square$

Overview



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{M}\mathbb{i}^{\mathcal{U}}_\kappa$	$\mathbb{M}\mathbb{L}^b_\kappa$
unbounded	✓	✓	✓				
eventually different	✗	✓	✓				
dominating	✗	✓	✗*				
unded	✗	?	?				
b-unbounded	✓	✓	✓				
b-eventually different	✗	✗	✓				
b-dominating	✗	✗	✓				

$(*)$ κ is weakly compact

The κ-**Eventually Different forcing** $\mathbb{E}_\kappa$ has conditions (s, F) where $s \in {}^{<\kappa}\kappa$ and $F \in [{}^{\kappa}\kappa]^{<\kappa}$. The ordering is defined by $(t, G) \leq (s, F)$ iff $s \subseteq t$ and $F \subseteq G$ and for $\alpha \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$ we have $t(\alpha) \notin \{f(\alpha) \mid f \in F\}$.

$\mathbb{E}_\kappa$ is $<\kappa$-closed and (κ, κ)-centred, and thus does not add a b-eventually different κ-real.

$\mathbb{E}_\kappa$ adds a κ-**Eventually Different real** $\bigcup\{s \mid (s, \cdot) \in G\}$, where G is $\mathbb{E}_\kappa$-generic. A κ-Eventually Different real f is unded over $\mathbf{V}$. Moreover, $\mathbb{E}_\kappa$ adds a κ-Cohen real, and thus a b-unbounded κ-real over $\mathbf{V}$.

A topological space X is **$<\kappa$-compact** if for every family of open sets C such that $X = \bigcup C$ there exists some $C' \in [C]^{<\kappa}$ such that $\bigcup C' = X$.

For a family $\langle X_i \mid i \in I \rangle$ of spaces, we define the **$<\kappa$-box topology** on the product $X = \prod_{i \in I} X_i$ as the topology generated by basic opens $[s] = \{f \in X \mid s \subseteq f\}$ for $s \in \prod_{i \in I'} X_i$ with $I' \in [I]^{<\kappa}$.

For strongly compact κ we can generalise Tychonoff's theorem: the product of $<\kappa$-compact spaces with the $<\kappa$-box topology is $<\kappa$-compact.

Theorem *Theorem 5.1 of Buhagiar and Džamonja [2021]*

κ is weakly compact iff for every family $\{X_i \mid i \in I\}$ with $|I| \leq \kappa$ and each X_i a $<\kappa$-compact space with $w(X_i) \leq \kappa$, the $<\kappa$-box product of $\{X_i \mid i \in I\}$ is $<\kappa$-compact.

Claim *Cf. [Miller, 1981, Lemma 5.1] for the ${}^{\omega}\omega$ analogue*

Assume κ is weakly compact. Let $\dot{x}$ be a $\mathbb{E}_\kappa$-name for a set in $\mathbf{V}$, let $s \in {}^{<\kappa}\kappa$ and $\lambda \in \kappa$, then there exists a set $\mathcal{X}$ with $|\mathcal{X}| < \kappa$ such that for all $F \in [{}^{\kappa}\kappa]^{\lambda}$ there is $p \leq (s, F)$ such that $p \Vdash$ "$\dot{x} \in \mathcal{X}$".

Proof. Give κ the cobounded topology, then it is $<\kappa$-compact and $w(\kappa) = \kappa$. Give ${}^{\kappa}\kappa$ and ${}^{\lambda\times\kappa}\kappa$ with $\lambda < \kappa$ the $<\kappa$-box topology, then these are $<\kappa$-compact by the weak Tychonoff theorem.

We conflate $F \in {}^{\lambda}({}^{\kappa}\kappa)$ with $\mathrm{ran}(F) \in [{}^{\kappa}\kappa]^{\lambda}$. For $X \subseteq \mathbf{V}$ define:

$$\mathcal{F}_X = \{F \in {}^{\lambda}({}^{\kappa}\kappa) \mid \exists p \in \mathbb{E}_\kappa (p \leq (s, F) \text{ and } p \Vdash \text{"} \dot{x} \in X \text{"})\}$$

Every $F \in {}^{\lambda}({}^{\kappa}\kappa)$ has a $y \in \mathbf{V}$ with $F \in \mathcal{F}_{\{y\}}$ and each $\mathcal{F}_X$ is open. Hence ${}^{\lambda}({}^{\kappa}\kappa) = \bigcup_{y\in\mathcal{Y}} \mathcal{F}_{\{y\}}$ and this has a subcover $\mathcal{X} \in [\mathcal{Y}]^{<\kappa}$. Note that $\mathcal{F}_{X\cup X'} \supseteq \mathcal{F}_X \cup \mathcal{F}_{X'}$, hence $F_{\mathcal{X}} = {}^{\lambda}({}^{\kappa}\kappa)$. □

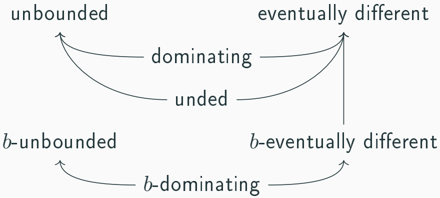
Lemma *Cf. [Miller, 1981, §5] for the ${}^\omega\omega$ analogue*

If κ is weakly compact, then $\mathbb{E}_\kappa$ does not add dominating reals.

Proof. Let $\langle (s_\eta, \lambda_\eta) \mid \eta \in \kappa\rangle$ list all (s, λ) with $s \in {}^{<\kappa}\kappa$ and $\lambda < \kappa$ such that each $(s, \lambda) = (s_\eta, \lambda_\eta)$ for κ many $\eta \in \kappa$.

Let $\Vdash_{\mathbb{E}}$ "$\dot{f} \in {}^\kappa\kappa$". Given $\eta \in \kappa$, by the claim, there exists $X_\eta \in [\kappa]^{<\kappa}$ such that for all $F \in [{}^\kappa\kappa]^{\lambda_\eta}$ there exists $p \leq (s_\eta, F)$ such that $p \Vdash$ "$\dot{f}(\eta) \in X_\eta$". We let $g : \eta \mapsto \sup(X_\eta) + 1$.

Let $(s, F) \in \mathbb{E}_\kappa$ and $\eta_0 \in \kappa$, then there exists $\eta \geq \eta_0$ such that $(s_\eta, \lambda_\eta) = (s, |F|)$. By the claim there exists $p \leq (s, F)$ such that $p \Vdash$ "$\dot{f}(\eta) \in X_\eta$" and thus $p \Vdash$ "$\dot{f}(\eta) < g(\eta)$". Since (s, F) and η_0 are arbitrary, we see that $\Vdash_{\mathbb{E}}$ "$g \not\leq^* \dot{f}$", thus $\dot{f}$ does not name a dominating κ-real. $\square$



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{Mi}^{\mathcal{U}}_\kappa$	$\mathbb{ML}^b_\kappa$
unbounded	✓	✓	✓	✓			
eventually different	✗	✓	✓	✓			
dominating	✗	✓	✗*	✗*			
unded	✗	?	?	✓			
b-unbounded	✓	✓	✓	✓			
b-eventually different	✗	✗	✓	✗			
b-dominating	✗	✗	✓	✗			

(∗) κ is weakly compact

Let $\mathcal{U}$ be a filter on κ. The κ-**Laver forcing** $\mathbb{L}_\kappa^\mathcal{U}$ guided by $\mathcal{U}$ has as conditions trees $T \subseteq {}^{<\kappa}\kappa$ where:

(i) T has a **stem** $s_T \in T$, i.e. s_T is the smallest splitting node.

(ii) For each $t \in T$ with $t \supseteq s_T$, the set of succesors of t are in $\mathcal{U}$.

(iii) If $t \in {}^{<\kappa}\kappa$ has limit height and $t \restriction \xi \in T$ for all $\xi \in \mathrm{dom}(t)$, then $t \in T$.

We order $\mathbb{L}_\kappa^\mathcal{U}$ by $S \leq T$ iff $S \subseteq T$.

$\mathbb{L}_\kappa^\mathcal{U}$ is $<\kappa$-closed if $\mathcal{U}$ is $<\kappa$-complete. $\mathbb{L}_\kappa^\mathcal{U}$ adds a dominating κ-real, but also a κ-Cohen real.

Question

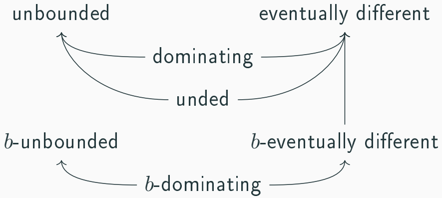
Does a dominating κ-real imply the existence of a κ-Cohen real?

Khomskii, Koelbing, Laguzzi, and Wohofsky [2022] showed that if $\mathbb{P} \subseteq \mathbb{L}_\kappa$ is a nontrivial subforcing and $(T)_s \in \mathbb{P}$ for all $T \in \mathbb{P}$ and $s \in T$, then $\mathbb{P}$ adds a κ-Cohen real.

Stronger yet, any $<\kappa$-distributive tree forcing for which $f(\dot{x})$ is a dominating κ-real, with $\dot{x}$ the generic κ-real and f a continuous function in the ground model, will add a κ-Cohen real.

The above question in general is still open.

Overview



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{Mi}^{\mathcal{U}}_\kappa$	$\mathbb{ML}^b_\kappa$
unbounded	✓	✓	✓	✓	✓		
eventually different	✗	✓	✓	✓	✓		
dominating	✗	✓	✗*	✗*	✓		
unded	✗	?	?	✓	?		
b-unbounded	✓	✓	✓	✓	✓		
b-eventually different	✗	✗	✓	✗	?		
b-dominating	✗	✗	✓	✗	?		

(∗) κ is weakly compact

Let $\mathcal{U}$ be a filter on κ. The κ-**Miller forcing** $\mathbb{Mi}_\kappa^{\mathcal{U}}$ guided by $\mathcal{U}$ has as conditions trees $T \subseteq {}^{<\kappa}\kappa$ where:

(i) For each $s \in T$ there exists $t \supseteq s$ that is splitting.

(ii) If $t \in T$ is splitting, the set of successors of t are in $\mathcal{U}$.

(iii) If $t \in {}^{<\kappa}\kappa$ has limit height and the set of $\xi \in \mathrm{dom}(t)$ such that $t \restriction \xi$ is splitting in T is cofinal in $\mathrm{dom}(t)$, then $t \in T$ and the set of successors of t is in $\mathcal{U}$.

We order $\mathbb{Mi}_\kappa^{\mathcal{U}}$ by $S \leq T$ iff $S \subseteq T$.

$\mathbb{Mi}_\kappa^{\mathcal{U}}$ is $<\kappa$-closed if $\mathcal{U}$ is $<\kappa$-complete and adds an unbounded κ-real.

A forcing $\mathbb{P}$ has the (b, h)-κ-**Laver property** if $\Vdash_{\mathbb{P}}$ " $\dot{f} \in \prod b$" implies that there exists a function φ with $\mathrm{dom}(\varphi) = \kappa$ such that $|\varphi(\alpha)| \leq h(\alpha)$ and $\Vdash_{\mathbb{P}}$ " $\dot{f}(\alpha) \in \varphi(\alpha)$" for each $\alpha \in \kappa$.

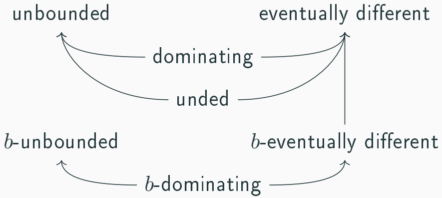
Let pow : $\alpha \mapsto 2^{|\alpha|}$.

Theorem *Proposition 81 in Brendle et al. [2018]*

If $\mathcal{U}$ is a $<\kappa$-complete ultrafilter, then $\mathbb{Mi}_\kappa^{\mathcal{U}}$ has the (b, pow)-κ-Laver property for every $b \in {}^\kappa\kappa$.

If $\text{pow}(\alpha) < \mathrm{cf}(b(\alpha))$ for each $\alpha \in \kappa$, then $\mathbb{Mi}_\kappa^{\mathcal{U}}$ does not add a b-unbounded κ-real.

Overview



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{Mi}^{\mathcal{U}}_\kappa$	$\mathbb{ML}^b_\kappa$
unbounded	✓	✓	✓	✓	✓	✓	
eventually different	✗	✓	✓	✓	✓	?	
dominating	✗	✓	✗*	✗*	✓	?	
unded	✗	?	?	✓	?	?	
b-unbounded	✓	✓	✓	✓	✓	✗**	
b-eventually different	✗	✗	✓	✗	?	?	
b-dominating	✗	✗	✓	✗	?	✗	

(∗) κ is weakly compact (∗∗) κ is measurable

The κ-**Miller Lite forcing** $\mathbb{ML}^b_\kappa$ guided by a function b has as conditions trees $T \subseteq \prod_{<\kappa} b$ where:

(i) For each $s \in T$ there exists $t \supseteq s$ that is splitting.

(ii) If $t \in T$ is splitting, the set of successors is equal to $b(\mathrm{dom}(t))$.

(iii) If $t \in {}^{<\kappa}\kappa$ has limit height and the set of $\xi \in \mathrm{dom}(t)$ such that $t \restriction \xi$ is splitting in T is cofinal in $\mathrm{dom}(t)$, then $t \in T$ and the set of successors of t is equal to $b(\mathrm{dom}(t))$

We order $\mathbb{ML}^b_\kappa$ by $S \leq T$ iff $S \subseteq T$.

Theorem *Lemma 1.3 in vdV. [2023]*

$\mathbb{ML}^b_\kappa$ is $<\kappa$-closed.

$\mathbb{ML}^b_\kappa$ adds a b-unbounded κ-real.

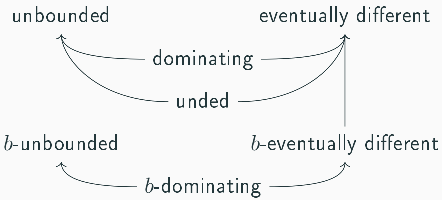
A forcing $\mathbb{P}$ has the h-κ-**Sacks property** if $\Vdash_{\mathbb{P}}$ " $\dot{f} \in {}^{\kappa}\kappa$" implies that there exists a function φ with $\mathrm{dom}(\varphi) = \kappa$ such that $|\varphi(\alpha)| \leq h(\alpha)$ and $\Vdash_{\mathbb{P}}$ " $\dot{f}(\alpha) \in \varphi(\alpha)$" for each $\alpha \in \kappa$.

If $\mathbb{P}$ has the h-κ-Sacks property, then $\mathbb{P}$ does not add an unbounded κ-real. Also, $\mathbb{P}$ then has the (b, h)-κ-Laver property for all $b \in {}^{\kappa}\kappa$.

Theorem *Theorem 1.8 in vdV. [2023]*

$\mathbb{ML}^b_\kappa$ has the h-κ-Sacks property for $h : \alpha \mapsto b(\alpha)^{|\alpha|}$.

$\mathbb{ML}^b_\kappa$ does not add an unbounded κ-real. If $h(\alpha) < \mathrm{cf}(b^*(\alpha))$ for each $\alpha \in \kappa$, then $\mathbb{ML}^b_\kappa$ does not add a b^*-unbounded κ-real.



	$\mathbb{C}_\kappa$	$\mathbb{D}_\kappa$	$\mathbb{D}^b_\kappa$	$\mathbb{E}_\kappa$	$\mathbb{L}^{\mathcal{U}}_\kappa$	$\mathbb{Mi}^{\mathcal{U}}_\kappa$	$\mathbb{ML}^b_\kappa$
unbounded	✓	✓	✓	✓	✓	✓	✗
eventually different	✗	✓	✓	✓	✓	?	?
dominating	✗	✓	✗*	✗*	✓	?	✗
unded	✗	?	?	✓	?	?	✗
b-unbounded	✓	✓	✓	✓	✓	✗**	✓
b-eventually different	✗	✗	✓	✗	?	?	?
b-dominating	✗	✗	✓	✗	?	✗	?

$(*)$ κ is weakly compact $(**)$ κ is measurable

The Next Step

Iteration: preservation of not adding witnesses.

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