Set Theories in Parallel Universes

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Outline

- Construction of Boolean-valued models of ZFC.

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- Construction of Boolean-valued models of ZFC.
- Generalized algebra-valued models.
- Non-classical algebra-valued models of a fragment of ZF and full ZF.
- Foundation of mathematics in a class of non-classical algebra-valued models.
- A new interpretation function.
- Independence set theoretic results in some non-classical set theories.

Construction of Boolean Valued Model

In the following steps, it will be discussed briefly that how a Boolean valued model is constructed and in which sense it becomes a model of ZFC. The whole construction will take place over a standard model $\mathbf{V}$ of ZFC.

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1. Let us take a complete Boolean algebra, $\mathbb{B} = \langle B, \wedge, \vee, \Rightarrow, ^*, 0, 1\rangle$.
2. For any ordinal α we define,

$$\mathbf{V}_{\alpha}^{(\mathbb{B})} = \{x : \mathrm{Func}(x) \wedge \mathrm{ran}(x) \subseteq B \wedge \exists \xi < \alpha(\mathrm{dom}(x) \subseteq \mathbf{V}_{\xi}^{(\mathbb{B})})\}$$

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3. Using the above we get a Boolean valued model as,

$$\mathbf{V}^{(\mathbb{B})} = \{x : \exists \alpha (x \in \mathbf{V}_{\alpha}^{(\mathbb{B})})\}$$

4. Extend the language of classical ZFC by adding a name corresponding to each element of $\mathbf{V}^{(\mathbb{B})}$, in it.
5. Associate every formula of the extended language with a value of B by the map $[\![.]\!]$. First we give the algebraic expressions which associate the two basic well-formed formulas with values of B. For any u, v in $\mathbf{V}^{(\mathbb{B})}$,

$$[\![u \in v]\!] = \bigvee_{x \in \mathrm{dom}(v)} (v(x) \wedge [\![x = u]\!])$$

$$[\![u = v]\!] = \bigwedge_{x \in \mathrm{dom}(u)} (u(x) \Rightarrow [\![x \in v]\!]) \wedge \bigwedge_{y \in \mathrm{dom}(v)} (v(y) \Rightarrow [\![y \in u]\!])$$

6. Then for any sentences σ and τ of the new language we define,

$$\llbracket \sigma \wedge \tau \rrbracket = \llbracket \sigma \rrbracket \wedge \llbracket \tau \rrbracket$$
$$\llbracket \sigma \vee \tau \rrbracket = \llbracket \sigma \rrbracket \vee \llbracket \tau \rrbracket$$
$$\llbracket \sigma \rightarrow \tau \rrbracket = \llbracket \sigma \rrbracket \Rightarrow \llbracket \tau \rrbracket$$
$$\llbracket \neg \sigma \rrbracket = \llbracket \sigma \rrbracket^{*}$$
$$\llbracket \forall x \varphi(x) \rrbracket = \bigwedge_{x \in \mathbf{V}^{(\mathbb{B})}} \llbracket \varphi(x) \rrbracket$$
$$\llbracket \exists x \varphi(x) \rrbracket = \bigvee_{x \in \mathbf{V}^{(\mathbb{B})}} \llbracket \varphi(x) \rrbracket$$

7. A sentence σ will be called *valid* in $\mathbf{V}^{(\mathbb{B})}$ or $\mathbf{V}^{(\mathbb{B})}$ will be called a model of a sentence σ if $\llbracket \sigma \rrbracket = 1$. It will be denoted as $\mathbf{V}^{(\mathbb{B})} \models \sigma$.

8. Then we have the following celebrated result:

Theorem

For any complete Boolean algebra $\mathbb{B}$, $\mathbf{V}^{(\mathbb{B})} \models \mathrm{ZFC}$, *i.e., all the classical logic axioms and* ZFC *axioms are valid in* $\mathbf{V}^{(\mathbb{B})}$.

Heyting-Valued Model

If the complete Boolean algebras are replaced by complete Heyting algebras $\mathbb{H}$ in the previous construction, then we get the following theorem.

Theorem

$\mathbf{V}^{(\mathbb{H})} \models \mathrm{IZF}$, *where* IZF *stands for the intuitionistic Zermelo-Fraenkel set theory: Zermelo-Fraenkel set theoretic axioms over the intuitionistic logic.*

Generalising the Concept

Let us consider an algebra $\mathbb{A} = \langle A, \wedge, \vee, \Rightarrow, ^*, 1, 0\rangle$, where $\langle A, \wedge, \vee, 1, 0\rangle$ is a complete distributive lattice.

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1. $1 \in D$ but $0 \notin D$,
2. if $a \in D$ and $a \leq b$, then $b \in D$,
3. for any two elements $a, b \in D$, $a \wedge b \in D$.

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Construct $\mathbf{V}^{(\mathbb{A})}$ as similar to the Boolean-valued models. Define an assignment function $[\![\cdot]\!]_{\mathbb{A}}$ (or simply denoted by $[\![\cdot]\!]$) from the collection of all sentences of the extended language into A. Fix a designated set D of $\mathbb{A}$.

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Definition

A sentence φ of the extended language is said to be valid in $\mathbf{V}^{(\mathbb{A})}$, denoted by $\mathbf{V}^{(\mathbb{A})} \models_D \varphi$ or by $\mathbf{V}^{(\mathbb{A})} \models \varphi$ (when the designated set is clear from the context), if $[\![\varphi]\!] \in D$.

Bounded Quantification Property, BQ_φ

Let us consider an algebra $\mathbb{A} = \langle A, \wedge, \vee, \Rightarrow, ^*, 1, 0\rangle$, where $\langle A, \wedge, \vee, 1, 0\rangle$ is a complete distributive lattice. For any formula $\varphi(x)$, we say that bounded quantification property of φ holds in $\mathbf{V}^{(\mathbb{A})}$ if for any $u \in \mathbf{V}^{(\mathbb{A})}$ the following holds:

$$[\![\forall x(x \in u \to \varphi(x))]\!] = \bigwedge_{x \in dom(u)} (u(x) \Rightarrow [\![\varphi(x)]\!]). \qquad (\mathcal{BQ}_\varphi)$$

Reasonable Implication Algebra

Definition

An algebra $\mathbb{A} = \langle A, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ is called a reasonable implication algebra if $\langle A, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a complete distributive lattice and $\Rightarrow$ has the following properties:

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P1: $x \wedge y \leq z$ implies $x \leq y \Rightarrow z$.

P2: $y \leq z$ implies $x \Rightarrow y \leq x \Rightarrow z$.

P3: $y \leq z$ implies $z \Rightarrow x \leq y \Rightarrow x$.

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A reasonable implication algebra is said to be *deductive* if it satisfies

$$(x \wedge y) \Rightarrow z = x \Rightarrow (y \Rightarrow z). \quad (P4)$$

Negation Free Fragment (NFF)

If $\mathcal{L}$ is any first-order language including the connectives $\wedge$, $\vee$, $\bot$, $\rightarrow$, and $\neg$ and Λ any class of $\mathcal{L}$-formulas, we denote closure of Λ under $\wedge$, $\vee$, $\bot$, $\rightarrow$, $\exists$, and $\forall$ by $\mathrm{Cl}(\Lambda)$ and call it the *negation-free closure of* Λ. A class Λ of formulas is *negation-free closed* if $\mathrm{Cl}(\Lambda) = \Lambda$. By NFF we denote the negation-free closure of the atomic formulas; its elements are called the negation-free formulas.

An Algebra-Valued Model of a Set Theory

Theorem

Let $\mathbb{A}$ be a deductive reasonable implication algebra such that $\mathbf{V}^{(\mathbb{A})}$ satisfies the NFF-bounded quantification property $(\mathrm{NFF}\text{-}\mathcal{BQ}_\varphi)$. Then Extensionality, Pairing, Infinity, Union, Power Set, NFF-Separation and NFF-Replacement are valid in $\mathbf{V}^{(\mathbb{A})}$, for any designated set D of $\mathbb{A}$.

NFF-(...) stands for the instances of (...) only for the negation free formulas.

(Löwe, B., and S. Tarafder, Generalized Algebra-Valued Models of Set Theory, *Review of Symbolic Logic*, Cambridge University Press, 8(1), pp. 192–205, 2015.)

Three Valued Matrix PS_3

Let us consider the three valued matrix $\mathrm{PS}_3 = \langle \{1, \nicefrac{1}{2}, 0\}, \wedge, \vee, \Rightarrow, ^*, 1, 0 \rangle$ where the truth tables for the operators are given below:

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$\wedge$	1	1/2	0
1	1	1/2	0
1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\Rightarrow$	1	1/2	0
1	1	1	0
1/2	1	1	0
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1/2	1/2	1/2	0
0	0	0	0

$\vee$	1	1/2	0
1	1	1	1
1/2	1	1/2	1/2
0	1	1/2	0

$\Rightarrow$	1	1/2	0
1	1	1	0
1/2	1	1	0
0	1	1	1

$*$	
1	0
1/2	1/2
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Fix the designated set as $D = \{1, 1/2\}$.

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Theorem

$\mathbf{V}^{(\mathrm{PS}_3)} \models_D \mathrm{NFF\text{-}ZF}$.

The Logic of PS_3

Theorem

The logic sound and complete with respect to (PS_3, D) *is a maximal paraconsistent logic.*

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Theorem

The logic sound and complete with respect to (PS_3, D) is a maximal paraconsistent logic.

- Paraconsistent logic: A logic is said to be paraconsistent if there are formulas φ and ψ such that ψ cannot be derived from $\{\varphi, \neg\varphi\}$.
- Maximal: The logic of (PS_3, D) is a proper fragment of the classical propositional logic (CPL). If a theorem φ of CPL, which is not a theorem of the logic of (PS_3, D), is added as an axiom of the logic of (PS_3, D), then the resultant logic will be CPL.

Tarafder, S., & Chakraborty, M. K., *A Paraconsistent Logic Obtained from an Algebra-Valued Model of Set Theory*. New Directions in Paraconsistent Logic, Springer Proceedings in Mathematics & Statistics, Vol. 152. New Delhi: Springer, pp. 165 – 183 (2016).

Hence we have the following theorem.

Theorem

$\mathbf{V}^{(\mathrm{PS}_3)}$ *is an algebra-valued model of a paraconsistent set theory.*

Extension of PS_3

Definition

By a totally-ordered implicative algebra we shall mean an algebra of the form $\mathbb{A} = \langle A, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$, which satisfies the following conditions:

1. $\langle A, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a bounded lattice having top and bottom elements $\mathbf{1}$ and $\mathbf{0}$, respectively,

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1. $\langle A, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a bounded lattice having top and bottom elements $\mathbf{1}$ and $\mathbf{0}$, respectively,
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2. the lattice order $\leq$ is a totally-ordered relation on A, and
3. the operator $\Rightarrow$ is defined by

$$a \Rightarrow b = \begin{cases} \mathbf{0}, & \text{if } a \neq \mathbf{0} \text{ and } b = \mathbf{0}; \\ \mathbf{1}, & \text{otherwise,} \end{cases} \qquad (\dagger)$$

for any two elements $a, b \in \mathbf{A}$.

Extension of PS_3

Lemma (Jockwich & Venturi)

Any totally-ordered implicative algebra $\mathbb{A}$ is a deductive reasonable implicative algebra and NFF-$\mathcal{BQ}_\varphi$ holds in $\mathbf{V}^{(\mathbb{A})}$.

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Theorem

For any totally-ordered implicative algebra $\mathbb{A}$ and any designated set D, $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{NFF\text{-}ZF}$.

Inclusion of Negation

Let us consider a totally-ordered implicative algebra $\mathbb{A}$ associated with a unary operator ^{*_1}, which is defined as follows: for any element a in $\mathbb{A}$

$$a^{*_1} = \begin{cases} \mathbf{0}, & \text{if } a = 1 \\ 1, & \text{if } a = 0 \\ a, & \text{otherwise.} \end{cases} \tag{$*_1$}$$

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It can be proved that for any designated set D $(\neq \{1\})$ of $\mathbb{A}$, the logic of $(\mathbb{A}, {}^{*_1}, D)$ is paraconsistent. Hence, $\mathbf{V}^{(\mathbb{A}, {}^{*_1})}$ becomes an algebra-valued model of a paraconsistent set theory which validates $\mathrm{NFF\text{-}ZF}$.

Inclusion of Negation

Let us now consider another unary operator ^{*_2} on a totally-ordered implicative algebra $\mathbb{A}$, which is defined as follows: for any element a in $\mathbb{A}$

$$a^{*_2} = \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{otherwise.} \end{cases} \tag{\$}$$

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$$a^{*_2} = \begin{cases} 1, & \text{if } a = 0 \\ 0, & \text{otherwise.} \end{cases} \qquad (\$)$$

Theorem

For any totally-ordered implicative algebra $\mathbb{A}$, associated with a unary operator ^{*_2}, as similar as defined in (\$), $\mathbf{V}^{(\mathbb{A}, {}^{*_2})} \models_D \mathrm{ZF}$, for any designated set D.

(Jockwich, S. and Venturi, G. (2021), Non-classical models of ZF. Studia Logica, 109, pp. 509 – 537).

Foundation of Mathematics

Definition

An algebra $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ is called a well-ordered deductive reasonable implication algebra if the following conditions hold:

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a bounded lattice having top and bottom elements $\mathbf{1}$ and $\mathbf{0}$, respectively,

(ii) the lattice order $\leq$ is a well-ordered relation on $\mathbf{A}$, and

(iii) the operator $\Rightarrow$ is defined as in $(\dagger)$.

$\mathcal{PS}$-Algebra

Definition

Let $\mathcal{PS}$ be the collection of all algebras $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$ satisfying the following conditions:

(i) $\langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ is a well-ordered deductive reasonable implication algebra,

(ii) * is a unary operator on $\mathbf{A}$ which satisfies $\mathbf{1}^* = \mathbf{0}$, $\mathbf{0}^* = \mathbf{1}$.

Any algebra $\mathbb{A} \in \mathcal{PS}$ will be called a $\mathcal{PS}$-algebra.

Logics corresponding to $\mathcal{PS}$-algebras

Depending on the operator * in a $\mathcal{PS}$-algebra $\mathbb{A}$ and the designated set D, it can be shown that the logic of $(\mathbb{A}, D)$ might be the classical logic or some non-classical logic (viz. paraconsistent, paracomplete, neither paraconsistent nor paracomplete).

$\mathcal{PS}$-Algebra-Valued Models

Combining all the above results we get that for any $\mathcal{PS}$-algebra $\mathbb{A}$ and a designated set D, $\mathbf{V}^{(\mathbb{A})}$ becomes an algebra-valued model of the classical set theory or some non-classical set theory. In particular,
$\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{NFF\text{-}ZF}$.

Let us consider the formula

$$\forall x \forall y(\mathrm{SetNat}(x) \wedge y \subseteq x \wedge \mathrm{Ind}(y) \rightarrow x = y), \tag{MI}$$

where $\mathrm{SetNat}(x)$ and $\mathrm{Ind}(y)$ are the first order formulas naively state that 'x is the set of all natural numbers' and 'y is an inductive set', respectively.

Mathematical Induction

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Theorem (Tarafder)

The formula MI is valid in any $\mathcal{PS}$-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$, i.e., $\mathbf{V}^{(\mathbb{A})} \models_D$ MI, corresponding to any designated set D of $\mathbb{A}$.

The Axiom of Choice

It is known that the first order formula defining the Axiom of Choice (AC) is given below:

$\forall u(\neg(u = \varnothing) \to \exists f(\mathrm{Func}(f) \wedge \mathrm{Dom}(f; u) \wedge \forall x(x \in u \wedge \neg(x = \varnothing) \to$
$\exists z \exists y(\mathrm{Pair}(z; x, y) \wedge z \in f \wedge y \in x)))),$

where the naive interpretations of the first order formulas $\mathrm{Func}(f), \mathrm{Dom}(f; u)$, and $\mathrm{Pair}(z; x, y)$ are respectively 'f is a function','domain of f is u', and 'z is the pair (x, y)'.

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Theorem (Tarafder)

If $\mathbf{V} \models \mathrm{AC}$, then for any $\mathcal{PS}$-algebra $\mathbb{A}$ and any ultra-designated set D of $\mathbb{A}$, $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{AC}$.

Cantor's Theorem and the Schröder–Bernstein Theorem

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Schröder–Bernstein Theorem

For any $\mathcal{PS}$-algebra $\mathbb{A}$ and designated set D, if $u, v \in \mathbf{V}^{(\mathbb{A})}$ are such that

$$\mathbf{V}^{(\mathbb{A})} \models_D \exists g \; \mathrm{InjFunc}(g; u, v) \wedge \exists h \; \mathrm{InjFunc}(h; v, u)$$

holds then $\mathbf{V}^{(\mathbb{A})} \models_D \exists f \; \mathrm{BijFunc}(f; u, v)$ also holds.

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holds then $\mathbf{V}^{(\mathbb{A})} \models_D \exists f \ \mathrm{BijFunc}(f; u, v)$ also holds.

Cantor's Theorem

Let $\mathbb{A}$ be any $\mathcal{PS}$-algebra, D be a designated set and $u \in \mathbf{V}^{(\mathbb{A})}$ be an arbitrary element. If $v \in \mathbf{V}^{(\mathbb{A})}$ is such that $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{Pow}(u, v)$, then $\mathbf{V}^{(\mathbb{A})} \models_D |u|_{\mathbf{V}^{(\mathbb{A})}} < |v|_{\mathbf{V}^{(\mathbb{A})}}$ also holds.

Generalized Continuum Hypothesis (GCH)

The naive statement of GCH is as follows.

For any infinite set s there does not exist any set t such that the cardinal number of s is less than the cardinal number of t and the cardinal number of t is less than the cardinal number of the power set of s.

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Theorem (Tarafder)

For any $\mathcal{PS}$-algebra $\mathbb{A}$, ultra-designated set D, and a model $\mathbf{V}$ of ZFC, $\mathbf{V} \models \mathrm{GCH}$ if and only if $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{GCH}$.

(Tarafder, S. (2022). Non-classical foundations of set theory. *The Journal of Symbolic Logic*, 87(1), pp. 347–376.)

Question: Are the Axiom of Choice, Zorn's Lemma (ZL) and Well-Ordering Theorem (WOT) equivalent in these non-classical algebra-valued models?

Cobounded-Algebra

An algebra $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ is called a Cobounded-algebra if

(i) $\langle \mathbf{A}, \wedge, \vee, \mathbf{1}, \mathbf{0} \rangle$ is a complete distributive lattice,

(ii) if $\bigvee_{i \in I} a_i = \mathbf{1}$, then there exists $j \in I$ such that $a_j = \mathbf{1}$, where I is an index set and $a_i \in A$ for all $i \in I$,

(iii) if $\bigwedge_{i \in I} a_i = \mathbf{0}$, then there exists $j \in I$ such that $a_j = \mathbf{0}$, where I is an index set and $a_i \in A$ for all $i \in I$,

(iv) the binary operator $\Rightarrow$ is defined as follows: for $a, b \in A$

$$a \Rightarrow b = \begin{cases} \mathbf{0}, & \text{if } a \neq \mathbf{0} \text{ and } b = \mathbf{0}; \\ \mathbf{1}, & \text{otherwise.} \end{cases}$$

Designated Cobounded-Algebra

Let $\mathbb{A} = \langle \mathbf{A}, \wedge, \vee, \Rightarrow, ^*, \mathbf{1}, \mathbf{0} \rangle$ be an algebra and D be a designated set of $\mathbb{A}$. The pair $(\mathbb{A}, D)$ is called a designated Cobounded-algebra if

(i) $\langle \mathbf{A}, \wedge, \vee, \Rightarrow, \mathbf{1}, \mathbf{0} \rangle$ is a Cobounded-algebra,

(ii) the operator * is defined as follows: for $a, b \in A$

$$a^* = \begin{cases} \mathbf{0}, & \text{if } a = \mathbf{1}; \\ a, & \text{if } a \in D \setminus \{\mathbf{1}\}; \\ \mathbf{1}, & \text{if } a \notin D. \end{cases}$$

A designated Cobounded-algebra $(\mathbb{A}, D)$ is called an ultra-designated Cobounded-algebra, if the corresponding designated set D is an ultrafilter.

Cobounded-Algebra-Valued Models

All the foundational results shown in $\mathcal{PS}$-algebra-valued models hold in the Cobounded-algebra-valued models as well. Moreover, the Transfinite Induction is valid in these models.

Equivalence of the Axiom of Choice

Theorem (Ganguly & Tarafder, 2025)

For any Cobounded-algebra $\mathbb{A}$ and the ultradesignated set D, the following are equivalent.

(i) $\mathbf{V} \models \mathrm{AC}$

(ii) $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{AC}$

(iii) $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{ZL}$

(iv) $\mathbf{V}^{(\mathbb{A})} \models_D \mathrm{WOT}$

A Drawback

Leibniz's law of indiscernibility of identicals:

$$\forall x \forall y \big((x = y \land \varphi(x)) \rightarrow \varphi(y) \big) \qquad (\mathsf{LL}_\varphi)$$

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$$\forall x \forall y \big((x = y \wedge \varphi(x)) \rightarrow \varphi(y) \big) \qquad (\mathsf{LL}_\varphi)$$

It can be proved that there are formulas φ so that LL_φ fails to be valid in any totally-ordered-algebra-valued models.

A Rectification

To rectify the issue regarding LL_φ, we have modified the assignment function $[\![\cdot]\!]$. In other words, we have modified the algebraic interpretations of $=$ and $\in$ to rectify the problem. To distinguish the standard assignment function and the modified one, we denote the former by $[\![\cdot]\!]_{\mathrm{BA}}$ and the later by $[\![\cdot]\!]_{\mathrm{PA}}$.

Cobounded-Algebra-Valued Models Under a New Interpretation Function

Let us consider a designated Cobounded-algebra $(\mathbb{A}, D)$ and the following assignment function on the cobounded-algebra-valued model $\mathbf{V}^{(\mathbb{A})}$.

$$
\begin{aligned}
[\![u \in v]\!]_{\mathrm{PA}} &= \bigvee_{x \in \mathrm{dom}(v)} \big(v(x) \wedge [\![x = u]\!]_{\mathrm{PA}}\big), \\
[\![u = v]\!]_{\mathrm{PA}} &= \bigwedge_{x \in \mathrm{dom}(u)} \big((u(x) \Rightarrow [\![x \in v]\!]_{\mathrm{PA}}) \wedge ([\![x \in v]\!]^{*}_{\mathrm{PA}} \Rightarrow u(x)^{*})\big) \\
&\quad \wedge \bigwedge_{y \in \mathrm{dom}(v)} \big((v(y) \Rightarrow [\![y \in u]\!]_{\mathrm{PA}}) \wedge ([\![y \in u]\!]^{*}_{\mathrm{PA}} \Rightarrow v(y)^{*})\big);
\end{aligned}
$$

Then extend the assignment function homomorphically, similar to the Boolean-valued models.

$\overline{\textit{Extensionality}}$

One of the axioms of ZF, Axiom of Extensionality is modified together with the interpretation function. We name the modified version $\overline{\text{Extensionality}}$, which is as follows:

$$\forall x \forall y \forall z \big(((z \in x \leftrightarrow z \in y) \wedge (\neg(z \in x) \leftrightarrow \neg(z \in y))) \rightarrow x = y \big).$$

Extensionality

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Let $\overline{\mathrm{ZF}}$ be the axiom system containing all the axioms of ZF where Extensionality is replaced by $\overline{\text{Extensionality}}$. Notice that if the basic logic is classical, then ZF and $\overline{\mathrm{ZF}}$ are two equivalent systems.

Comparison

	Axioms of ZF	Axioms of $\overline{\mathrm{ZF}}$	LL_φ
$[\![\cdot]\!]_{\mathrm{BA}}$	Valid	Valid	Valid
$[\![\cdot]\!]_{\mathrm{PA}}$	Valid	Valid	Valid

Table: With respect to Boolean-valued models, $\mathbf{V}^{(\mathbb{A})}$.

	Axioms of ZF	Axioms of $\overline{\mathrm{ZF}}$	LL_φ
$[\![\cdot]\!]_{\mathrm{BA}}$	NFF-ZF is valid	NFF-ZF + $\overline{\text{Extensionality}}$ is valid	Fails for some φ
$[\![\cdot]\!]_{\mathrm{PA}}$	ZF − Extensionality is valid	Valid	Valid

Table: With respect to ultra-designated Cobounded-algebra valued models, $\mathbf{V}^{(\mathbb{A})}$, where $\mathbb{A}$ has more than two elements.

(Jockwich, S., Tarafder, S. and Venturi, G. (2024). ZF and its interpretations, Annals of Pure and Applied Logic, 175(6): 103427.

Independence results

Definition

Let T and φ be respectively a theory and a sentence in the language of ZFC. We say that φ is independent from T whenever there are two models $\mathcal{M}_1$ and $\mathcal{M}_2$ such that:

1. $\mathcal{M}_1 \models \mathsf{T}$ and $\mathcal{M}_2 \models \mathsf{T}$,
2. $\mathcal{M}_1 \models \varphi$,
3. $\mathcal{M}_2 \not\models \varphi$.

Hereditary Independence

If a formula φ is independent from ZF then φ is independent from any fragment T of ZF.

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Question: Can we find two models $\mathcal{M}_1$ and $\mathcal{M}_2$ of a given proper fragment T of ZF, which are not models of ZF, such that $\mathcal{M}_1 \models \varphi$ but $\mathcal{M}_2 \not\models \varphi$?

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Theorem

Let φ be a sentence in the language of ZF, T be a proper fragment of ZF, and $\mathbb{A}$ be a complete algebra such that

1. *φ is independent with respect to ZF,*
2. *the logic of $\mathbb{A}$ is a proper fragment of the classical propositional logic,*
3. $\mathbf{V}^{(\mathbb{A})} \models \varphi$ *and* $\mathbf{V}^{(\mathbb{A})} \models \mathrm{T}$.

Then, there are two algebra-valued models of T, but not of ZF, which do not agree on the validity of φ.

Application

Definition

Let $\mathbb{A}$ be a complete algebra. Then, by $\mathbb{A}$-ZF we mean the fragment of ZF which is valid in all algebra-valued models of the form $\mathbf{V}^{(\mathbb{A}\times\mathbb{B})}$, for all complete Boolean algebra $\mathbb{B}$.

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Observation

We can prove that:

1. NFF-ZF is included in PS_3-ZF,
2. PS_3-ZF is a proper fragment of ZF, and
3. PS_3-ZF is a paraconsistent set theory.

Application

Theorem

There are two algebra-valued models of PS_3-ZF*, and not of* ZF*, which do not agree on the validity of the Continuum Hypothesis, thus showing the independence of the Continuum Hypothesis from* PS_3-ZF*.*

Independence in the Fragment Only

Theorem

If $\mathbb{A}$ is a complete algebra and φ is a sentence in the language of ZF, such that one of the following two (exclusive) conditions holds:

1. $\mathbf{V}^{(\mathbb{A})} \models \varphi$ *but* $\mathrm{ZF} \models \neg\varphi$,
2. $\mathbf{V}^{(\mathbb{A})} \not\models \varphi$ *but* $\mathrm{ZF} \models \varphi$.

then, φ is independent from $\mathbb{A}$-ZF but not from ZF.

Application

Example

Let us consider the following formula

$$\forall x \exists y \forall z \big(z \in y \leftrightarrow (z \in x \wedge (\neg \exists w (w \in z))) \big). \qquad \text{(Sep)}$$

The formula Sep is an instance of the Separation Axiom. In addition, $\mathbf{V}^{(\mathrm{PS}_3)} \not\models$ Sep. Hence, Sep is independent from PS_3-ZF but not from ZF.

Application

Example

Consider the three-valued Heyting algebra $\mathbb{H}_3$. Let φ be the sentence which intuitively states that 'if κ is the cardinal number of a set, then 2^κ is the cardinal number of its power set'. It is well-known that, in IZF, the cardinality of the power set of a singleton set cannot be 2 (since this would imply the Law of Excluded Middle). Using this fact, we can prove that $\mathbf{V}^{(\mathbb{H}_3)} \not\models \varphi$. However, we know that $\mathrm{ZF} \models \varphi$. Hence, φ is independent from $\mathbb{H}_3$-ZF, but not from ZF.

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Tarafder, S. and Venturi, G. (2023). Independence proofs in non-classical set theories, The Review of Symbolic Logic, 16(4), pp. 979 – 1010.

Thank You...