Some logical aspects of topos theory and examples from algebraic geometry (Part II)

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June 29, 2020

Recap of Part I

- sheaves $\mathcal{F} \in Sh(X) \rightsquigarrow$ $\mathcal{F}(U)$ sets
  $$\begin{array}{c} \mathcal{F}(U) \\ \downarrow \\ \mathcal{F}(V) \end{array} \quad V \subseteq U$$
- $Sh(X)$ is a topos
  $f : \mathcal{F} \to \mathcal{G}$
  $R \subseteq \mathcal{F} \times \mathcal{G}$
- internal language:

$U \models s = s' \quad :\Leftrightarrow \quad s = s' \qquad s, s' \in \mathcal{F}(U)$

$U \models \phi \vee \psi \quad :\Leftrightarrow \quad$ there is $U = \bigcup_i U_i$ and $U_i \models \phi$ or $U_i \models \psi$

$U \models \bot \quad :\Leftrightarrow \quad U = \emptyset$

$U \models \phi \Rightarrow \psi \quad :\Leftrightarrow \quad (V \models \phi) \Rightarrow (V \models \psi)$ f.a. $V \subseteq U$

$$\neg\phi \equiv \underline{\phi \Rightarrow \bot}$$

$$\underbrace{\left(\bigcup_{U \models \phi} U\right)}_{[\![\phi]\!]} \models \phi$$

$$U \models \phi \Rightarrow \bot \quad \text{iff} \quad \underbrace{(V \models \phi) \Rightarrow V = \emptyset}_{[\![\phi]\!] \cap U = \emptyset}$$

$$U \models (\phi \Rightarrow \bot) \Rightarrow \bot$$

$$\text{iff} \quad \underbrace{V \models (\phi \Rightarrow \bot)}_{[\![\phi]\!] \cap V = \emptyset} \Rightarrow V = \emptyset$$

iff $[\![\phi]\!]$ is $\underset{\text{open}}{\smile}$ dense in U

Interpreting IZF in $\mathrm{Sh}(X)$

problem: $x \in y$ only makes sense for

$x : \mathcal{F}, \; y : \mathcal{P}(\mathcal{F})$

$$\bigcup_\alpha \mathcal{P}^\alpha(\emptyset) =: V$$

$$\text{“}\in\text{”} \subseteq V \times V$$

$$V_\alpha := \mathcal{P}^\alpha(\emptyset) \qquad \text{“}\in\text{”} \subseteq V_\alpha \times V_{\alpha+1}$$

Classifying toposes

Definition

A *geometric formula* is a first order formula ϕ that only uses the logical connectives

$$=, \top, \wedge, \bot, \vee, \exists, \bigvee.$$

$\cancel{\forall}$ $\cancel{\Rightarrow}$ $\cancel{\forall}$

A *geometric theory* is a first-order theory with axioms

$$\phi \vdash_{\vec{x}:\vec{A}} \psi$$

"$\forall x_1 : A_1 \ldots \forall x_n : A_n . \; \phi \Rightarrow \psi$"

where ϕ and ψ are geometric formulas.

$\top \vdash_{x,y,z:A} x \cdot (y \cdot z) = (x \cdot y) \cdot z$

$x = 0 \vee \exists y : A . \, xy = 1$

$x \neq 0 \Rightarrow \ldots$

Why geometric logic?

$\phi \wedge \psi$

$$\begin{tikzcd} S_1 \wedge S_1 \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\lrcorner"] & S_1 \arrow[d, hook] \\ S_2 \arrow[r, hook] & S \end{tikzcd}$$

Every continuous map $f : X \to Y$ induces

$$\mathrm{Sh}(X) \xleftarrow{f^*} \mathrm{Sh}(Y).$$

This f^* preserves finite limits and all colimits.
$\Rightarrow$ f^* preserves models of geometric theories.

In fact, every *geometric morphism* $f : \mathcal{E}' \to \mathcal{E}$ has such a "pull-back part"

$$\mathcal{E}' \xleftarrow{f^*} \mathcal{E}.$$

Definition

A geometric theory $\mathbb{T}$ is *classified* by a topos $\mathcal{E}_{\mathbb{T}}$ if there is a *universal model* $U_{\mathbb{T}} \in \mathrm{Mod}_{\mathcal{E}_{\mathbb{T}}}(\mathbb{T})$, that is:

$$\begin{array}{ccc} \mathsf{Geom}(\mathcal{E}', \mathcal{E}_{\mathbb{T}}) & \cong & \mathrm{Mod}_{\mathcal{E}'}(\mathbb{T}) \\ f & \mapsto & f^* U_{\mathbb{T}} \end{array}$$

is an $\underbrace{\text{equivalence of categories}}_{\text{think: bijection}}$ for every topos $\mathcal{E}'$.

Aside:

A *point* of $\mathcal{E}$ is a geometric morphism $\underset{\substack{\parallel \\ Sh(pt)}}{\mathsf{Set}} \to \mathcal{E}$.

points of $\mathcal{E}_{\mathbb{T}}$ = Geom $(\mathsf{Set}, \mathcal{E}_{\mathbb{T}})$ = $\mathrm{Mod}_{\mathsf{Set}}(\mathbb{T})$

$p \vee q$

$\neg(p \wedge q)$

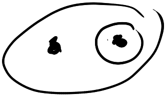
Example

What is the classifying topos of sub-singletons?

- one sort A
- no function/relation symbols
- one axiom: $\top \vdash_{x,y:A} x = y$

$p \vee \neg p$

$$\mathrm{Cont}(X, Y) = \{\text{opens } U \subseteq X\}$$

$Y =$ 

Grothendieck topos

Theorem

Every geometric theory has a unique (up to equivalence) classifying topos.
Every topos classifies some geometric theory.

Theorem

A geometric sequent $\phi \vdash_{\vec{x}:\vec{A}} \psi$ is fulfilled for $U_{\mathbb{T}}$ iff it is provable modulo $\mathbb{T}$.

Examples from algebraic geometry

Definition

A *site* is a category together with a “notion of covering” (a *Grothendieck topology*).

	site for $\mathrm{Sh}(X)$	site for Zar
objects	$U \subseteq X$ open	finitely presented ring A
morphisms	$V \subseteq U$	ring homomorphism $B \leftarrow A$
covers	$U = \bigcup_i U_i$	$(A[a_i^{-1}] \leftarrow A)_{i=1\dots n}$ whenever $a_1 + \ldots + a_n = 1$

Theorem

$\mathbb{A}^1$ is the universal local ring.
(So Zar classifies local rings.)

Proof sketch.

Infinitesimals? No problem!

$$\varepsilon \neq 0, \quad \underline{\varepsilon^2 = 0}$$

$$\mathbb{C}[X]/(X^2) \ni X$$

$$\operatorname{Spec} \mathbb{C} = \bullet$$

$$\operatorname{Spec} \mathbb{C}[X]/(X^2) = \bullet$$

Theorem

The big infinitesimal topos classifies the theory of infinitesimally thickened local rings.

~~Proof sketch.~~

- theory of rings: $A, \ldots$
- A local
- $\mathfrak{a} \subseteq A$ ideal

  with $x \in \mathfrak{a} \vdash_{x;A} \bigvee_{n \in \mathbb{N}} x^n = 0$