Some logical aspects of topos theory and examples from algebraic geometry

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Definition

An *(elementary) topos* is a category with

- finite limits and
- power objects.

Example

For a topological space X, the category $\mathrm{Sh}(X)$ of *sheaves* on X is a topos.

Sheaves

Definition

A *sheaf (of sets)* $\mathcal{F}$ on a topological space X is the following data:

- a set $\mathcal{F}(U)$ for every open $U \subseteq X$
- "restriction" maps $\mathcal{F}(U) \to \mathcal{F}(V)$ for $V \subseteq U$ such that $\mathcal{F}(U) \to \mathcal{F}(V) \to \mathcal{F}(W)$ is $\mathcal{F}(U) \to \mathcal{F}(W)$ for $W \subseteq V \subseteq U$

satisfying a certain *glueing condition*.

Examples

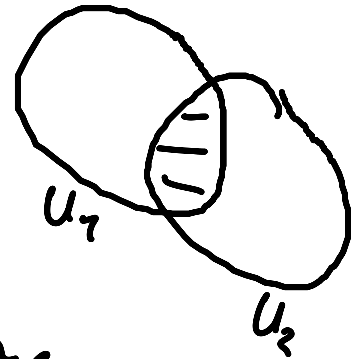
- $\mathcal{F}(U) = C^0(U) = C^0(U, \mathbb{R})$
- $C^0(\,\cdot\,, Y)$
- $C^\infty(\,\cdot\,, \mathbb{R})$ if X is a smooth manifold

- $\mathcal{F}(U) = \{U \to \mathbb{R}\}$

- $\mathcal{F}(U) = M$, M fixed

Glueing condition:

$\mathcal{F}(U_1 \cup U_2) = \mathcal{F}(U_1) \times \mathcal{F}(U_2)$ for $U_1 \cap U_2 = \varnothing$.



$\mathcal{F}(U_1 \cup U_2) = \mathcal{F}(U_1) \times_{\mathcal{F}(U_1 \cap U_2)} \mathcal{F}(U_2)$ $= \{(s \in \mathcal{F}(U_1), s' \in \mathcal{F}(U_2)) \mid s|_{U_1 \cap U_2} = s'|_{\dots}\}$

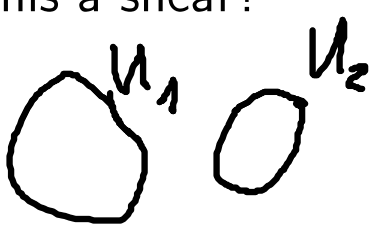
General requirement:
For $U = \bigcup_{i \in I} U_i$ and $s_i \in \mathcal{F}(U_i)$ with $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j, there is a unique $s \in \mathcal{F}(U)$ with $s|_{U_i} = s_i$ for all i.

$I = \varnothing$ $\quad$ $U = \varnothing$

$\Rightarrow |\mathcal{F}(\varnothing)| = 1$

Example

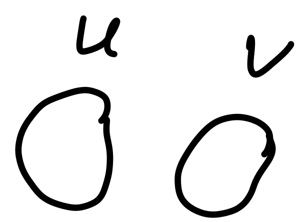
$\mathcal{F}(U) = M$ for a fixed set M.
Is this a sheaf?



$M = M \times M$

Example

For every set M we have the *constant sheaf* $\underline{M}(U) = \{\text{locally constant functions } U \to M\}$.



Remark

For every sheaf $\mathcal{F}$ we have $|\mathcal{F}(\emptyset)| = 1$. ✓

Remark

A sheaf on $X = \{pt\}$ is just a set.

$\mathcal{F}(\emptyset) = \{*\}$

$\mathcal{F}(\{pt\})$ arbitrary

$Sh(\{pt\})$ is the category of sets

$$\text{topological spaces} \hookrightarrow \text{toposes}$$
$$X \longmapsto Sh(X)$$

Internal language

We want to treat sheaves like sets/sorts/types.

$$\forall x : \mathcal{F}.\ \exists y : \mathcal{G}.\ \ldots$$

Want to get only statements that *can be checked locally*.

for $s, s' \in \mathcal{F}(X)$, $s = s'$ can be checked locally

for $f, g : X \to \mathbb{R}$, $f = g$ "

$\exists s \in \mathcal{F}(X)$ can not "

for $f : X \to \mathbb{R}$, f bounded can not "

$f : \mathcal{F} \to \mathcal{G}$

recursive definition (Kripke–Joyal semantics)

can contain terms $x : \mathcal{F}$ if $x \in \mathcal{F}(U)$

$U \models \phi \wedge \psi$ iff $U \models \phi$ and $U \models \psi$

$U \models \phi \vee \psi$ iff there is an open cover $U = \bigcup_{i \in I} U_i$ such that for every i, $U_i \models \phi$ or $U_i \models \psi$

$U \models f = 0 \vee f = 1$

$(f : U \to \mathbb{R})$

$U \models s = s'$ iff $s = s'$ in $\mathcal{F}(U)$

$(s, s' \in \mathcal{F}(U))$

$0, 1 : A$

$+, \cdot : A \times A \longrightarrow A$

Definition

A *sheaf of rings* on X is just a ring internal to $\mathrm{Sh}(X)$.

Example

C^0 is a ring internal to $\mathrm{Sh}(X)$.

$+, \cdot : C^0 \times C^0 \to C^0$

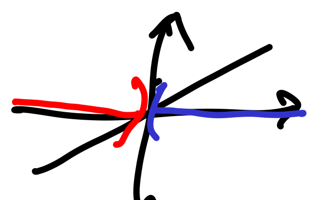
$C^0(U)$ is a ring

$\mathcal{O}_X$ - modules

example: C^0 looks like $\mathbb{R}$

$$f, g \in C^0(U), \qquad f < g\,?$$

$$f \in C^0(U), \qquad \overbrace{f<0} \vee \overbrace{f=0}^{\text{true on } \emptyset} \vee \overbrace{f>0}\,?\ \text{No!}$$



$$\mathcal{P}(\mathbb{Q}) \times \mathcal{P}(\mathbb{Q})$$

$$f < 1 \vee f > 0$$

c,

$$C^0 \cong \mathbb{R} = \{ (\underset{\underset{\mathcal{P}(\mathbb{Q})}{\shortparallel}}{L}, R) \mid \forall x, y \in \mathbb{Q}.\ (x<y \wedge y \in L \Rightarrow x \in L) \quad \ldots \}, \qquad L \subseteq \mathbb{Q},\ R \subseteq \mathbb{Q}$$

$$\underline{\mathbb{N}} \twoheadrightarrow \underline{\mathbb{Q}} \qquad\qquad \underline{\mathbb{R}} \subsetneq C^0$$